Problem 1. Let \((R, \mathfrak{m})\) be a 0-dimensional Noetherian (i.e. Artinian) local ring of prime characteristic \(p\) and \(M\) be a finitely generated \(R\)-module. Show that there exists \(e_0 \in \mathbb{N}\) such that 
\[ F^e(M) \cong F^{e_0}(M) \]
for all \(e \geq e_0\). (Actually, this is true as long as \(R\) is 0-dimensional Noetherian, the proof of which reduces to local case.)

Proof. Write down an representation \(R^m \xrightarrow{(a_{ij})} R^n \xrightarrow{} M \xrightarrow{} 0\) such that \(a_{ij} \in \mathfrak{m}\). Then, for any \(q = p^e\), \(F^e(M)\) is represented by exact sequence \(R^m \xrightarrow{(a_{ij}^e)} R^n \xrightarrow{} F^e(M) \xrightarrow{} 0\). By assumption, there exists \(q_0 = p^{e_0}\) such that \(m^{[q]} \equiv 0\) for all \(q \geq q_0\). Therefore \(F^e(M) \cong R^n\) for all \(e \geq e_0\).

(In case \(R\) is Artinian (not necessarily local), \(R = \prod_{i=1}^s R_i\) is a direct product of Artinian local rings \(R_i\). So the claim remains true although the stabled \(F^e(M)\) is not necessarily free over \(R\). Instead, \(F^e(M)\) is isomorphic to \(\oplus_{i=1}^s R_i^{n_i}\) for all \(e \gg 0\).)

Problem 2. Let \(R\) be a Noetherian ring of prime characteristic \(p\). It is known that \(I^* \subseteq \sqrt{I}\) for every ideal \(I \subseteq R\) (by comparing tight closure with integral closure). Here we show this from definition of tight closure.

(1) If \((R, \mathfrak{m})\) is local (with maximal ideal \(\mathfrak{m}\)), show \(\mathfrak{m}^* = \mathfrak{m}\).

(2) Show \(P^* = P\) for any prime ideal of \(R\), which is not assumed to be local.

(3) Show \(I^* \subseteq \sqrt{I}\) for every ideal \(I\) of \(R\).

(4) Compute \(0^*_R\). Here \(0\) refers to the zero ideal of \(R\).

Proof. (1). Suppose \(\mathfrak{m}^* \supseteq \mathfrak{m}\). Then \(1 \in \mathfrak{m}^*\), meaning that there exists \(c \in R^e\) such that \(c1^q \in \mathfrak{m}^{[q]}\) for all \(q \gg 0\). But this forces \(c \in \cap_{q \gg 0} \mathfrak{m}^{[q]} = 0\), a contradiction.

(2). By (1), the ideal \(P\) is tightly closed in \(R_P\) for any \(P \in \text{Spec}(R)\). This gives the conclusion \(P^* = P\) as \((P^*_R)R_P \subseteq (P_P)^*_R\) for any \(P \in \text{Spec}(R)\).

(3). For any ideal \(I\) of \(R\), say \(\sqrt{I} = \cap_{i=1}^s P_i\) for \(P_i \in \text{Spec}(R)\). Then \(I^* \subseteq \cap_{i=1}^s P_i^* = \cap_{i=1}^s P_i = \sqrt{I}\).

(4). We always have \(\sqrt{0} \subseteq 0^*\). By (3), we also have \(0^* \subseteq \sqrt{0}\). Thus \(0^* = \sqrt{0}\). \(\square\)

Problem 3. Let \(k\) be a field of characteristic 2, \(S = k[X, Y]\) be a polynomial ring over \(k\) with indeterminates \(X, Y\), and \(R = S/(X^3)S\). Let \(I = (X^2)R \subset (X)R = J\) be ideals of \(R\). (Then \(I \subset J\) are also modules over \(S\) under the natural ring homomorphism \(S \rightarrow R\).)

(1) Let \(F^e_R(-)\) be the Frobenius endomorphism over \(R\). Up to isomorphism, how many distinct \(R\)-modules are there among \(0, I, J, R, F_R(I), F_R(J), I^2_R, J^2_R, I^2_R, J^2_R\)? (Everything is considered as an \(R\)-module, including \(I\) as in the notation \(F_R(I)\), for example. Group isomorphic \(R\)-modules together.

(2) Determine \(I^*_R\) and \(I^*_J\) over \(R\). (Everything is considered as an \(R\)-module.)

(3) Determine \(I^*_R\) and \(I^*_J\) over \(S\). (Everything is considered as an \(S\)-module, including \(R\) as in \(I^*_R\).)

Proof. Denote the images of \(X, Y\) in \(R\) by \(x, y\). Therefore \(x^3 = 0 \in R\).

(1). As \(R\)-modules, \(I = xJ \cong R/xR\) and \(J \cong R/x^2R\). Hence \(F_R(I) \cong R/x^2R, F_R(J) \cong R/x^4R = R\) and \(I^2_R = x^2F_R(J) \cong x^2R = I\). We also have \(I^2_R = x^4R = 0\) and \(J^2_R = x^2R = I\). To sum up, there are four (4) distinct \(R\)-modules up to isomorphism, namely \(0 \cong I^2_R, I \cong J^2_R \cong I^2_R, J \cong F_R(I)\) and \(R \cong F_R(J)\).

(2). First, \(I^*_R = \sqrt{0} = J\) because \(I \subseteq \sqrt{0}\). Then notice that \(J^*_R = J = \sqrt{0}\), which means that \(0^*_R/J = 0\). But \(R/J \cong J/I\). So \(0^*_R/J = 0\), meaning \(I^*_J = J\).

(3). Over regular ring \(S, N_M = N\) for any \(S\)-modules \(N \subseteq M\). So \(I^*_R = I\) and \(I^*_J = I\) over \(S\). \(\square\)

Problem 4. Let \(R\) be a Noetherian ring of prime characteristic \(p\). Suppose

\[
G_* : \quad 0 \rightarrow G_n \xrightarrow{\phi_n} G_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_2} G_1 \xrightarrow{\phi_1} G_0
\]
is a complex with $G_i$ free of finite rank for $0 \leq i \leq n$. Apply Frobenius functor $F^e$ to get $F^e(G_i)$ for every $e \in \mathbb{N}$.

(1) If $G_\bullet$ is exact, show $F^e(G_\bullet)$ remains exact for every $e \in \mathbb{N}$.

(2) If $\ell_R(H_i(F^e(G_\bullet))) < \infty$ for every $1 \leq i \leq n$, show $\ell_R(H_i(F^e(G_\bullet))) < \infty$ for every $1 \leq i \leq n$ and for every $e \in \mathbb{N}$. (Here $\ell_R(–)$ represents the length of an $R$-module.)

Do (1) and (2) still hold if, instead, we assume $G_i$ are finitely generated projective $R$-modules for $0 \leq i \leq n$?

Proof. (1). Since Frobenius function $F^e$ commutes with localization and a complex is exact if and only if every localisation of it is exact, we may assume $R$ is local. Then the assumption that $G_\bullet$ is exact implies that depth$_{(\phi_i)}(R) \geq i$ and rank($G_i$) = rank($\phi_i$) + rank($\phi_{i+1}$) for every $1 \leq i \leq n$. In particular, $I(\phi_i)$ contains a non-zero-divisor for every $1 \leq i \leq n$, which guarantees that rank($\phi_i$) = rank($F^e(\phi_i)$) for every $i$ and every $e$. This also implies that $I(F^e(\phi_i)) = (I(\phi_i))^e$ and thus depth$_{(I(F^e(\phi_i))}(R) = \text{depth}_{I(\phi_i)}(R)$ for every $q = p^e$. Therefore, the complex $F^e(G_\bullet)$ still satisfies the ‘rank and depth’ condition. Consequently, $F^e(G_\bullet)$ is exact for every $e \in \mathbb{N}$. (If, for any given $i$, $\phi_i$ is represented by a matrix $(a_{jk})$, then $F^e(\phi_i)$ may be represented by the matrix $(a_{jk}^e)$.)

(2). This follows from (1). Indeed, for any given $e$, $\ell_R(H_i(F^e(G_\bullet))) < \infty$ for all $1 \leq i \leq n$ if and only if $F^e(G_\bullet) \otimes_R R_P$ is exact for all prime ideals $P$ that are not maximal.

Finally, (1) and (2) still hold if we assume $G_i$ are finitely generated projective $R$-modules for $0 \leq i \leq n$. As seen in the above proof, both (1) and (2) reduce to local case. $\square$

Problem 5. Let $R$ be a Noetherian ring of prime characteristic $p$ and $N \subseteq M$ be $R$-modules. Define $N^F_M = \{x \in M \mid x^q \in N^q_M \subseteq F^e(M) \text{ for some } q = p^e\}$. ($N^F_M$ is called the Frobenius closure of $N$ in $M$.)

1. Show that $N^F_M \subseteq N^e_M$.
2. Show that Frobenius closure commutes with localization, i.e. $(W^{-1}N)^F_{W^{-1}M} = W^{-1}(N^F_M)$ for any multiplicatively closed set $W \subseteq R$.

Proof. (It is routine to check that $N^F_M$ is an $R$-submodule of $M$.)

1. For any $x \in N^F_M$, we have $x^q \in N^q_M \subseteq F^e(M)$ for some $q = p^e$. This actually implies that $x^q \in N^q_M \subseteq F^e(M)$ for all $q \geq q_0$, showing $x \in N^e_M$ (with $c = 1 \in R^e$).

2. Let $W \subseteq R$ be a multiplicatively closed subset of $R$. If $x \in N^F_M$, then $x^q \in N^q_M \subseteq F^e(M)$ for some $q = p^e$. This gives $(x/1)^q \in (W^{-1}N)^q_{W^{-1}M} \subseteq F^e(W^{-1}M)$. (Here we use the fact that Frobenius functor commutes with localization.) Hence $x/1 \in (W^{-1}N)^F_{W^{-1}M}$, showing $(W^{-1}N)^F_{W^{-1}M} \supseteq W^{-1}(N^F_M)$. On the other hand, for any $x/w \in (W^{-1}N)^F_{W^{-1}M}$ with $x \in M$ and $w \in W$, there exists $q = p^e$ such that $(x/w)^q \in (W^{-1}N^q_{W^{-1}M} \subseteq F^e(W^{-1}M)$. (We may simply assume $w = 1$ as Frobenius closure is a submodule.) Using the fact that Frobenius functor commutes (naturally) with localization, we may write $(x/w)^q \in (W^{-1}N^q_{W^{-1}M} \subseteq W^{-1}(F^e(M))$ for the same $q = p^e$. Therefore there exists $w' \in W$ such that $w'x^q \in N^q_M \subseteq F^e(M)$ and hence $(w'x)^q \in N^q_M \subseteq F^e(M)$, which implies $w'x \in N^F_M$. This shows that $(W^{-1}N)^F_{W^{-1}M} \subseteq W^{-1}(N^F_M)$. Finally, $(W^{-1}N)^F_{W^{-1}M} = W^{-1}(N^F_M)$. $\square$