Problem 1. Let $R$ be a Noetherian ring of prime characteristic $p$. Show that $R$ has a weak test element if and only if $R/\sqrt{p}$ has a weak test element. (Here a weak test element is, by definition, a $q$-weak test element for some $q$.)

Proof. If $c \in R^o$ is a $q_1$-weak test element for $R$, then it is straightforward to verify that $c + \sqrt{0}$ is a $q_1$-weak test element for $R/\sqrt{0}$. Conversely, suppose $R/\sqrt{0}$ has a $q_2$-weak test element, say $d + \sqrt{0} \in (R/\sqrt{0})^o$ so that $d \in R^o$. Say $\sqrt{0^{[q]}} = 0$. Then direct checking shows that $d^{q_2}$, which is in $R^o$, is a $(q_2 q_3)$-weak test element for $R$. □

Problem 2. Let $R$ be a Noetherian ring of prime characteristic $p$ and $M$ an $R$-module. Recall that $^e M$ is the derived $R$-module structure on $M$ via the Frobenius homomorphism $F^e: R \rightarrow R$.

(1) If $^e M$ is a faithful $R$-module for some $e_0 > 0$, then $R$ is reduced and $^e M$ (including $M = 0^e_M$) are faithful for all $e \in \mathbb{N}$.

(2) Show that $\text{Ass}_R(M) = \text{Ass}_R(^e M)$ for every $e \in \mathbb{N}$.

Proof. (1). Denote $q_0 = p^{e_0}$, which is $\geq p$. Suppose $R$ is not reduced. Then there exists $0 \neq x \in \sqrt{0}$ such that $x^{q_0} = 0$. Then we see that $x \in \text{Ann}_R(^e M)$, a contradiction. Now that $R$ is reduced, the claim that $^e M$ is faithful for all $e$ follows immediately from the easy assertion that, quite generally, $\text{Ann}_R(^e M) \subseteq \text{Ann}_R(^{e_1} M) \subseteq \sqrt{\text{Ann}_R(^{e_2} M)}$ for every $e_1 \leq e_2$ and any $R$-module $M$.

(2). Firstly, we observe an easy claim that $\text{Ann}_R(x \in M) \subseteq \text{Ann}_R(x \in ^e M) \subseteq \sqrt{\text{Ann}_R(x \in M)}$ for any $x \in M$ and any $e \in \mathbb{N}$. Then for any $P \in \text{Ass}_R(M)$, there is $y \in M$ such that $\text{Ann}_R(y \in M) = P$. Hence $\text{Ann}_R(y \in ^e M) = P$ and therefore $P \in \text{Ass}_R(^e M)$. Conversely, suppose $P \in \text{Ass}_R(^e M)$, i.e. there is $z$ such that $\text{Ann}_R(z \in ^e M) = P$. Let $Rz$ be the $R$-submodule of $M$ generated by $z$. Then $P \in \text{Ass}(R/\text{Ann}_R(z \in M)) = \text{Ass}(Rz) \subseteq \text{Ass}_R(Rz) \subseteq \text{Ass}_R(M)$. □

Problem 3. Let $(R, m, k)$ be a Noetherian equidimensional catenary local ring of prime characteristic $p$ with dim$(R) = d$. Suppose $q^d < \ell_R(R/m^{[q]}) < q^d + q$ for some $q = p^e \geq p$. Prove $\text{Sing}(R) = \{m\}$, where $\text{Sing}(R) = \{P \in \text{Spec}(R) \mid R_P$ is not regular$\}$ is the singular locus of $R$.

Proof. The assumption of $(R, m, k)$ being equidimensional catenary guarantees that dim$(R/P) + \text{dim}(R_P) = \text{dim}(R)$ for every $P \in \text{Spec}(R)$. And $R$ is not regular as $q^d < \ell_R(R/m^{[q]})$.

If dim$(R) = 0$, then there is nothing to prove. So we assume dim$(R) > 1$ and it suffices to show $R_P$ is regular for any prime ideal $P$ such that dim$(R/P) = 1$ (and hence dim$(R_P) = d - 1$). For any such $P$, $\ell_R(R_P/P_P^{[q]}) \leq \frac{1}{q} \ell_R(R/m^{[q]}) < q^{d-1} + 1 = q^{\text{dim}(R_P)} + 1$, which implies $R_P$ is regular. □

Problem 4. Let $R$ be a ring (not necessarily of characteristic $p$). Given $R$-modules $M, N$ and $f \in \text{Hom}_R(M, N)$, we say $f$ is pure if the induced map $f \otimes_R 1_L: M \otimes_R L \rightarrow N \otimes_R L$ is injective for every $R$-module $L$. (Denote by m-Spec$(R)$ the set consisting of all maximal ideals of $R$.)

(1) If $f \in \text{Hom}_R(M, N)$ is pure, then $f$ is injective. (Therefore, $f \in \text{Hom}_R(M, N)$ is pure if and only if $f$ is injective and the inclusion map $f(M) \subseteq N$ is pure.)

(2) $f \in \text{Hom}_R(M, N)$ is pure if and only if $f_P: M_P \rightarrow N_P$ is pure for every $P \in \text{Spec}(R)$ if and only if $f_m: M_m \rightarrow N_m$ is pure for every $m \in \text{m-Spec}(R)$.

(3) Show (A) $f \in \text{Hom}_R(M, N)$ is pure if and only if $f \otimes_R 1_L: M \otimes_R L \rightarrow N \otimes_R L$ is injective for every finitely generated $R$-module $L$; and (B) If $R$ is Noetherian and $M$ is finitely generated, then $f \in \text{Hom}_R(M, N)$ is pure if and only if $f \otimes_R 1_L: M \otimes_R L \rightarrow N \otimes_R L$ is injective for every finitely generated $R$-module $L$ such that $\text{Ass}_R(L) = \{m\}$ for some $m \in \text{m-Spec}(R)$.

(4) Suppose $R$ is Noetherian and $M, N$ are finitely generated $R$-modules. Then $f \in \text{Hom}_R(M, N)$ is pure if and only if $f_m: M_m \rightarrow N_m$ is pure for every $m \in \text{m-Spec}(R)$ if and only if $f: M \rightarrow N$ splits (meaning there exists $g \in \text{Hom}_R(N, M)$ such that $g \circ f = 1_M$).

(5) Suppose $R$ is Noetherian and $F$ is a free $R$-module. Then $f \in \text{Hom}_R(F, N)$ is pure if and only if the induced map $f \otimes_R 1_E: F \otimes_R E \rightarrow N \otimes_R E$ is injective, where $E = \bigoplus_{m \in \text{m-Spec}(R)} E_R(R/m)$. 

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Proof. (1). This follows from the injectivity of \( f \otimes_R 1_R : M \otimes_R R \to N \otimes_R R \).

(2). This is standard.

(3). In both (A) and (B), we only need to show ‘if’. Suppose \( f : M \to N \) is not pure. Then there exists an \( R \)-module \( L \) such that \( f \otimes_R 1_L \) has a non-zero kernel. Then, by property of tensor product, there exists a (sufficiently large) finitely generated \( R \)-submodule \( L' \subseteq L \) such that \( f \otimes_R 1_{L'} \) is not injective, which proves (A). If, moreover, \( R \) is Noetherian and \( M \) is finitely generated, then \( 0 \neq \ker(f \otimes_R 1_{L'}) \subseteq M \otimes_R L' \) are all finitely generated \( R \)-modules. Choose \( m \in \text{m-Spec}(R) \) such that \( 0 \neq (\ker(f \otimes_R 1_{L'}))_m \). Then, by Krull intersection theorem, Artin-Rees Lemma etc., \( \ker(f \otimes_R 1_{L'/m^nL'}) \neq 0 \) for some integer \( n \gg 0 \), which proves case (B) as \( \text{Ass}_R(L'/m^nL') = \{m\} \).

(4). Without loss of generality, we assume \((R, m, k)\) is local. Denote \(-\rightarrow = \text{Hom}_R(-, E_R(k))\). Then \( f \) is pure \( f \otimes_R L \) is injective for all \( R \)-module \( L \) such that \( \ell_R(L) < \infty \) \( f \otimes_R L \) is injective for all \( R \)-module \( L \) such that \( \ell_R(L) < \infty \) \( f \) is pure \( f \otimes_R \hat{M}^\vee : \hat{M} \otimes_R \hat{M}^\vee \to \hat{N} \otimes_R \hat{M}^\vee \) is injective \( (f \otimes_R \hat{M}^\vee)\vee : (\hat{N} \otimes_R \hat{M}^\vee)\vee \to (\hat{M} \otimes_R \hat{M}^\vee)\vee \) is surjective \( \text{Hom}_R(f, \hat{M}) : \text{Hom}_R(\hat{N}, \hat{M}) \to \text{Hom}_R(\hat{M}, \hat{M}) \) is surjective \( f \) splits \( f \) splits \( f \) is pure.

(5). Without loss of generality, we assume \((R, m, k)\) is local. We only need to show ‘if’. Suppose \( f \in \text{Hom}_R(F, N) \) is not pure. Then, as \( F \) is free (not necessarily of finite rank), an argument similar to the one in part (3) above shows \( \ker(f \otimes_R 1_L) \neq 0 \) for some \( L \) with \( \ell(L) < \infty \). Then there exists an integer \( n > 0 \) such that \( L \) is embedded into \( E^m \) where \( E = E_R(k) \). Then, as \( F \) is free, we have \( \ker(f \otimes_R 1_{E^m}) \neq 0 \) \( \ker(f \otimes_R 1_E) \neq 0 \), a contradiction. \( \square \)

Problem 5. Given a local Noetherian ring \((R, m, k)\) of prime characteristic \( p \) (not necessarily \( F \)-finite), one could define \( R \) to be strongly \( F \)-regular if, for any \( c \in R^p \), there exists an integer \( e \geq 1 \) such that the \( R \)-linear map \( R \to R^e \) sending 1 to \( c \) is pure. In general, one could define \( R \) is strongly \( F \)-regular if \( R_m \) is strongly \( F \)-regular for every \( m \in \text{m-Spec}(R) \). (By Problem 4, we see that the above definition agrees with the one given in class when \( R \) is \( F \)-finite.)

(1) If there exists a pure \( R \)-linear map \( R \to \hat{R} \) sending 1 to \( c \) with \( e \geq 1 \), then \( R \) is reduced and, for every \( e' \geq e \), the \( R \)-linear map \( R \to \hat{R}^e \) sending 1 to \( c \) is pure. (Thus the above definition of strong \( F \)-regularity forces \( R \) to be reduced.)

(2) Show that \((R, m, k)\) is strongly \( F \)-regular if and only if \( 0_{E_R(k)} = 0 \).

Proof. (1). The given pure map shows \( \hat{R} \) is faithful, implying \( R \) is reduced by Problem 2(1). So we are free to identify \( \hat{R} \) with \( R^{1/q} \) as \( R \)-modules for any \( q = p^e \). Thus the given pure map may be considered as \( f : R \to R^{1/q} \) sending 1 to \( c^{1/q} \). But \( f = g \circ i : R \subseteq R^{1/p} \to R^{1/q} \) in which \( g \) is the \( R^{1/p} \)-linear map sending 1 to \( c^{1/q} \). Thus the purity of \( f \) forces the purity of inclusion map \( i \) (which is easy to check). Also the purity of \( f \) amounts to the purity of the \( R^{1/p} \)-linear map \( f' : R^{1/p} \to R^{1/q} \) sending 1 to \( c^{1/q} \), which readily implies the purity of \( f' \) as an \( R \)-linear map. Therefore the \( R \)-linear map \( f' \circ i : R \to R^{1/q} \) is pure (which is easy to check) and it sends 1 to \( c^{1/q} \). In other words, the \( R \)-linear map \( R \to R^{1/q} \) sending 1 to \( c \) is pure. This in enough to prove (1).

(2). First of all, for any \( c \in R \) and \( e \in \mathbb{N} \), let us denote by \( f_{c,e} : R \to \hat{R} \) the \( R \)-linear map sending 1 to \( c \in \hat{R} \).

To show ‘only if’, suppose \( R \) is strongly \( F \)-regular. For any \( x \in 0_{E_R(k)}^e \), by the definition of tight closure, there exists \( c \in R^e \) such that \( 0 = c \otimes_R x \in \hat{R} \otimes_R E_R(k) \) for all \( e \gg 0 \). Thus \( 1 \otimes_R x \in R \otimes_R E_R(k) \) is in \( \ker(f_{c,x} \otimes_R 1_R) \) for all \( e \gg 0 \). But, by part (1) above, we know that \( f_{c,x} : R \to \hat{R} \) are pure for all \( e \gg 0 \), which forces \( 0 = 1 \otimes_R x \in R \otimes_R E_R(k) \), implying \( x = 0 \). So \( 0_{E_R(k)}^e = 0 \).

Finally, let us prove ‘if’. Choose \( 0 \neq w \in (0 :_{E_R(k)} m) \) so that \( w \) generates the socle of \( E_R(k) \). The assumption \( 0_{E_R(k)}^e = 0 \) implies that \( w \notin 0_{E_R(k)}^e \). Thus, for any \( c \in R^o \), there exists an integer \( e \geq 1 \) such that \( 0 \neq c \otimes_R w \in \hat{R} \otimes_R E_R(k) \), which means that \( 1 \otimes_R w \in R \otimes_R E_R(k) \) is not in \( \ker(f_{c,x} \otimes_R 1_R) \), which implies that \( \ker(f_{c,x} \otimes_R 1_R) = 0 \) as every non-zero \( R \)-submodule of \( R \otimes_R E_R(k) \) contains \( 1 \otimes_R w \). Thus \( f_{c,x} \otimes_R 1_R : R \otimes_R E_R(k) \to \hat{R} \otimes_R E_R(k) \) is injective and, therefore, \( f_{c,x} : R \to \hat{R} \) is pure by Problem 4(5). Hence \( R \) is strongly \( F \)-regular. \( \square \)