Problem 1. Let $R$ be a Noetherian ring of characteristic $p$, $I$ an ideal, and $M$ an $R$-module. 
(1) For any given $e \in \mathbb{N}$, show that $H^i_I(\mathcal{O}^e M) \cong \mathfrak{q}(H^i_I(M))$ for every $i$. 
(2) Suppose that $\mathcal{O}^e M$ is finitely generated over $R$ for some $e_0 \geq 1$. Show that $R/\text{Ann}_R(M)$ is an $F$-finite ring. Consequently, $\mathcal{O}$ is finitely generated over $R$ for every $e$.

Problem 2. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of prime characteristic $p$ and $P \in \text{Spec}(R)$ be any prime ideal of $R$. Suppose $R$ is $F$-finite and say $[k : k^p] = p^n$.

(1) Prove that $\dim(R/Q) = \dim(R/P)$ for every $Q \in \text{Ass}_R(R/P) = \min(R/R\widehat{P})$.

(2) Show that $[(R/P)_P : ((R/P)_P)^p] = p^{\alpha + \dim(R/P)}$. (We have proved this when $R$ is complete.)

Problem 3. Let $R$ be a Noetherian ring of prime characteristic $p$, $M$ a finitely generated $R$-module such that $\text{Ann}_R(M) \subseteq \sqrt{0}$. For any $R$-modules $N \subseteq L$ and $x \in L$, prove $x \in N^p$ if and only if there exists $c \in R^p$ such that $\text{Image}(x \otimes R \mathcal{O} M) \subseteq \text{Image}(N \otimes R \mathcal{O} M \to L \otimes R \mathcal{O} M)$ for all $e \gg 0$.

Problem 4. Let $R$ be a Noetherian $F$-finite ring of prime characteristic $p$, $M$ a finitely generated $R$-module with FFRT by finitely generated $R$-modules $M_1, M_2, \ldots, M_r$ and $L$ a finitely generated $R$-module. Show that $\cup_{e \in \mathbb{N}} \text{Ass}(L \otimes R \mathcal{O} M)$ is a finite set and, moreover, there exists an integer $k \in \mathbb{N}$ such that the following are satisfied.

(1) For every $e \in \mathbb{N}$, there exists a primary decomposition
$$0 = Q_{e_1} \cap Q_{e_2} \cap \cdots \cap Q_{e_s}$$
in $L \otimes R \mathcal{O} M$,
where $\text{Ass}(L \otimes \mathcal{O} M) = \{P_{e_j} \mid 1 \leq j \leq s_e\}$ and $Q_{e_j}$ are $P_{e_j}$-primary components of $0 \subseteq L \otimes R \mathcal{O} M$ satisfying $P_{e_j}(L \otimes R \mathcal{O} M) \subseteq Q_{e_j}$ for all $1 \leq j \leq s_e$.

(2) We have $J^k(0_{L \otimes R \mathcal{O} M}) = 0$, i.e., $J^k H^0_\mathfrak{m}(L \otimes R \mathcal{O} M) = 0$ for all $J \subseteq R$ and for all $e \in \mathbb{N}$.
(In case $L = R/I$, the above may be stated in terms of $\cup_{e \in \mathbb{N}} \text{Ass}(M/I^{[e]} M)$ and $H^0_\mathfrak{m}(M/I^{[e]} M)$.)

Problem 5. Let $R \to S$ be a homomorphism of Noetherian rings of prime characteristic $p$ and $M$ a finitely generated $R$-module. (In this problem, we treat $\mathcal{O}$ as an $R$-$R$-bimodule where $r_1 \cdot x \cdot r_2 = r_1^e r_2 x$ for any $r_1, r_2 \in R, x \in M$. Also recall that $#_R(\mathcal{O} M) = \ell_R(\text{Image}(k \otimes_R \mathcal{O} M \xrightarrow{\psi = \id} E_R(k) \otimes_R \mathcal{O} M))$ where $\psi : k \to E_R(k)$ is any injective $R$-map and $\ell_R(-)$ denotes length as a right $R$-module.)

(1) For any $R$-module $E$, there is an isomorphism $(E \otimes \mathcal{O} M) \otimes_R S \cong (E \otimes_R S) \otimes_S (M \otimes_R S)$.

Moreover, the isomorphism is natural in the sense that for any $R$-linear map $E_1 \to E_2$, the following diagram commutes:

$$
\begin{array}{ccc}
(E_1 \otimes \mathcal{O} M) \otimes_R S & \xrightarrow{\cong} & (E_1 \otimes_R S) \otimes_S (M \otimes_R S) \\
\downarrow & & \downarrow \\
(E_2 \otimes \mathcal{O} M) \otimes_R S & \xrightarrow{\cong} & (E_2 \otimes_R S) \otimes_S (M \otimes_R S)
\end{array}
$$

(2) Assume, furthermore, that $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ is a flat homomorphism of local rings such that $\mathfrak{m} S = \mathfrak{n}$. Show that (a) $E_R(k) \otimes S \cong E_S(l)$ and (b) $#_R(\mathcal{O} M) = #_S(\mathcal{O} M \otimes_R S)$ for all $e$. 

Due: 04/18/05 (Mon.)