The linear growth property of $\text{Tor}^R_C(M/I^m M, N/J^n N)$

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1. Primary decompositions

2. The linear growth property

3. The linear growth property of \( \{ \text{Tor}^R_c(M/IM, N/J^nN) \} \)
Basic notations

Throughout this talk, the following notations are used

- Let $R$ be a commutative Noetherian ring.
- Let $M$ and $N$ be finitely generated $R$-modules.
- Let $I_1, \ldots, I_s$ and $J_1, \ldots, J_t$ be ideals of $R$, where $s, t \in \mathbb{N}$.
- For $\underline{m} := (m_1, \ldots, m_s) \in \mathbb{N}^s$ and $\underline{n} := (n_1, \ldots, n_t) \in \mathbb{N}^t$, denote
  
  $I^m := I_1^{m_1} \cdots I_s^{m_s}$, \hspace{1cm} |\underline{m}| := m_1 + \cdots + m_s$

  $J^n := J_1^{n_1} \cdots J_t^{n_t}$, \hspace{1cm} and \hspace{1cm} |\underline{n}| := n_1 + \cdots + n_t$
1 Primary decompositions

2 The linear growth property

3 The linear growth property of \( \{ \text{Tor}_C^R(M/IM, N/J^nN) \} \)
Definitions

**Definition**

The set of *associated primes* of an $R$-module $M$, denoted $\text{Ass}(M)$, is defined as follows

$$\text{Ass}(M) := \{ P \in \text{Spec}(R) \mid \exists x \in M \text{ such that } \text{Ann}(x) = P \}$$

It is known that $\text{Ass}(M) = \emptyset \iff M = 0$.

**Definition**

We say a submodule $Q$ is $P$-primary in $M$ if $\text{Ass}(M/Q) = \{P\}$.

Or equivalently, $Q$ is $P$-primary in $M \iff \sqrt{\text{Ann}(M/Q)} = P$ and every $r \in R \setminus P$ is a non-zerodivisor on $M/Q$. 
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Or equivalently, $Q$ is $P$-primary in $M \iff \sqrt{\text{Ann}(M/Q)} = P$ and every $r \in R \setminus P$ is a non-zerodivisor on $M/Q$. 
Existence of primary decompositions

Theorem (Noether)

Every submodule $K$ of $M$ has a primary decomposition, that is

$$K = Q_1 \cap \cdots \cap Q_r \quad \text{with } Q_i \text{ being } P_i\text{-primary for } i = 1, \ldots, r.$$ 

We may always assume that $P_1, \ldots, P_r$ are distinct and $\bigcap_{i \neq j} P_i \supsetneq K$ for all $j = 1, \ldots, r$, which implies $\text{Ass}(M/K) = \{P_1, \ldots, P_r\}$.

Remark

The primary decompositions of $K$ in $M$ are in one-one correspondence to the primary decompositions of 0 in $M/K$. Thus, there is no loss of generality in studying the primary decomposition of 0 in $M$. 
1. Primary decompositions

2. The linear growth property

3. The linear growth property of \( \{ \text{Tor}_c^R (M/\text{Im} M, N/\text{J}^n N) \} \)
Swanson proved the linear growth property of $R/I^m$. More generally, we have

**Theorem (Swanson, Sharp, Y)**

The linear growth property holds for primary decompositions of $I^m M$ in $M$, namely, there exists $k \in \mathbb{N}$ such that for every $m \in \mathbb{N}^s$, there exists a primary decomposition

$$I^m M = Q_{m,1} \cap \cdots \cap Q_{m,r(m)}$$

such that $P_{m,i}^k | m | M \subseteq Q_{m,i}$ for all $i = 1, \ldots, r(m)$.

Note that one could equivalently state the above theorem in terms of the primary decompositions of 0 in $M/I^m M$. 
Let \( \{M_m\}_{m \in \mathbb{N}^s} \) be a family of finitely generated \( R \)-modules.

**Definition**

We say \( \{M_m\}_{m \in \mathbb{N}^s} \) has the **linear growth property** if there exists \( k \in \mathbb{N} \) such that for every \( m \in \mathbb{N}^s \) (such that \( M_m \neq 0 \)), there exists a primary decomposition

\[
0 = Q_{m,1} \cap \cdots \cap Q_{m,r(m)} \quad \text{with } Q_{m,i} \text{ being } P_{m,i}-\text{primary in } M_m
\]

such that \( P_{m,i}^{k|m|} M_m \subseteq Q_{m,i} \) for all \( i = 1, \ldots, r(m) \).

Thus, in this terminology, the above theorem simply says that the family \( \{M / IM \}_{m \in \mathbb{N}^s} \) has the linear growth property.
Other known cases of linear growth:
\{\text{Tor}_c^R(N, M/ImM)\} and \{\text{Ext}_R^c(N, M/ImM)\}

In fact, the linear growth of \{M/ImM\}_{m \in \mathbb{N}^s} is contained in the following

**Theorem (Y)**

*Let c be a fixed integer. The linear growth property holds for the families \{\text{Tor}_c^R(N, M/ImM)\}_{m \in \mathbb{N}^s} and \{\text{Ext}_R^c(N, M/ImM)\}_{m \in \mathbb{N}^s}.*
More: \( \{ H_c(F_\cdot \otimes M/IM) \} \) and \( \{ H^c(\text{Hom}(F_\cdot, M/IM)) \} \)

Even more generally, we have

**Theorem (Y)**

Let \( R \) be an algebra over \( A \) and let

\[
F_\cdot : \cdots \rightarrow F_i \rightarrow F_{i-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F_{-1} \rightarrow \cdots
\]

be a complex of finitely generated projective \( A \)-modules. Then, for any integer \( c \), the families (of \( R \)-modules)

\[
\{ H_c(F_\cdot \otimes_A M/IM) \}_{m \in \mathbb{N}^s} \quad \text{and} \quad \{ H^c(\text{Hom}_A(F_\cdot, M/IM)) \}_{m \in \mathbb{N}^s}
\]

satisfy the linear growth property.
Primary decompositions

The linear growth property

The linear growth property of $\{\text{Tor}_c^R(M/IM, N/J^nN)\}$
Main result

Theorem (Y)

Let $c$ be a fixed integer. The linear growth property holds for the family
\[ \{ \text{Tor}^R_c(M/I^mM, N/J^nN) \}_{(m,n) \in \mathbb{N}^{s+t}}. \]

Remark

It is not hard to show that the linear growth property holds for
\[ \{ \text{Ext}^0_R(M/I^mM, N/J^nN) \}_{(m,n) \in \mathbb{N}^{s+t}}. \] however, it is not clear whether the linear growth property holds for
\[ \{ \text{Ext}^c_R(M/I^mM, N/J^nN) \}_{(m,n) \in \mathbb{N}^{s+t}} \] for a general $c$. 
Proof (in case $s = t = 1$)

For (notational) simplicity, we show a proof of the linear growth property of the family $\{ \text{Tor}_c^R(M/I^m M, N/J^n N)\}_{(m,n) \in \mathbb{N}^2}$.

Write $I = (a_1, \ldots, a_u)$ and define $\mathbb{Z}$-graded rings and modules

$$R_I := R[X_1, \ldots, X_u, X^{-1}] \quad \text{with} \quad \deg(X_i) = 1 \quad \text{and} \quad \deg(X^{-1}) = -1$$

$$R_I := R[IX, X^{-1}] = \bigoplus_{s \in \mathbb{Z}} I^s X^s$$

$$M_I := \bigoplus_{s \in \mathbb{Z}} I^s MX^s = \cdots \oplus MX^{-1} \oplus M \oplus IMX \oplus I^2 MX^2 \oplus \cdots$$

Then $M_I$ is naturally a finitely generated graded module over $R_I$. Note there is a surjective graded ring homomorphism $R_I \twoheadrightarrow R_I$ by $X_i \mapsto a_i X$ and $X^{-1} \mapsto X^{-1}$. Thus $M_I$ is a finitely generated graded module over $R_I$. 
Proof (continued)

Similarly, write $J = (b_1, \ldots, b_v)$ and define $\mathbb{Z}$-graded rings and modules

$$R_J := R[Y_1, \ldots, Y_v, Y^{-1}] \quad \text{with} \quad \deg(Y_j) = 1 \quad \text{and} \quad \deg(Y^{-1}) = -1$$

$$R_J := R[JY, Y^{-1}] = \bigoplus_{t \in \mathbb{Z}} J^t Y^t$$

$$N_J := \bigoplus_{t \in \mathbb{Z}} J^t NY^t = \cdots \oplus NY^{-1} \oplus N \oplus JNY \oplus J^2 NY^2 \oplus \cdots$$

Then $N_J$ is naturally a finitely generated graded module over $R_J$ and there is a surjective graded ring homomorphism $R_J \twoheadrightarrow R_J$ by $Y_j \mapsto b_j Y$ and $Y^{-1} \mapsto Y^{-1}$. Thus $N_J$ is a finitely generated graded module over $R_J$.
Proof (continued)

Note that $R_I = R[X_1, \ldots, X_u, X^{-1}]$ and $R_J = R[Y_1, \ldots, Y_v, Y^{-1}]$ are both polynomial rings over $R$. Thus each of the graded components is a free $R$-module.

Write down graded free resolutions of $\mathcal{M}_I$ over $R_I$ and $\mathcal{N}_J$ over $R_J$ respectively (by finite rank free modules)

\[
\begin{align*}
\mathcal{F}_* : & \quad \cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to \mathcal{M}_I \to 0 \\
\mathcal{G}_* : & \quad \cdots \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to \mathcal{N}_J \to 0
\end{align*}
\]

Then $\mathcal{F}_* \otimes_R \mathcal{G}_*$ is (naturally) a $\mathbb{Z}^2$-graded complex of finitely generated free modules over the $\mathbb{Z}^2$-graded polynomial ring

\[
R_I \otimes_R R_J \cong R[X_1, \ldots, X_u, X^{-1}; Y_1, \ldots, Y_v, Y^{-1}] =: R.
\]
Proof (continued)

Thus, by a previous theorem, the family

$$\left\{ H_c \left( ( \mathcal{F} \otimes_R \mathcal{G} ) \otimes_R \frac{\mathcal{R}}{(X_m Y_n)} \right) \right\}_{(m,n) \in \mathbb{N}^2}$$

has the linear growth property.

However, the above linear growth property implies the linear growth property of the family $$\{ \text{Tor}_c^R(M/I^m M, N/J^n N) \}_{(m,n) \in \mathbb{N}^2}.$$
Proof (continued)

First, for any \(m, n \in \mathbb{N}\), note

\[
(\mathcal{F} \otimes_R \mathcal{G}) \otimes_R \frac{R}{(X^{-m}Y^{-n})} \cong \left(\mathcal{F} \otimes_{R_1} \frac{R_1}{(X^{-m})}\right) \otimes_R \left(\mathcal{G} \otimes_{R_J} \frac{R_J}{(Y^{-n})}\right).
\]

Second, since \(X^{-1}\) is regular on both \(\mathcal{M}_I\) and \(\mathcal{R}_I\) while \(Y^{-1}\) is regular on both \(\mathcal{N}_J\) and \(\mathcal{R}_J\), we see that

\[
\mathcal{F} \otimes_{R_1} \frac{R_1}{(X^{-m})} \quad \text{and} \quad \mathcal{G} \otimes_{R_J} \frac{R_J}{(Y^{-n})}
\]

are graded free resolutions of \(\mathcal{M}_I/\mathcal{M}_I\) and \(\mathcal{N}_J/\mathcal{N}_J\) over graded rings \(R_1/\mathcal{M}_I\) and \(R_J/\mathcal{N}_J\) respectively.
Proof (continued)

Thus, for all \(m, n \in \mathbb{N}\),

- \(\left( \mathcal{F} \otimes R \frac{R}{(X-m)} \right)_0\) is an \(R\)-resolution of \((\mathcal{M}_i/X^{-m}\mathcal{M}_i)_0 = M/I^mM\).
- \(\left( \mathcal{G} \otimes R \frac{R}{(Y-n)} \right)_0\) is an \(R\)-resolution of \((\mathcal{N}_J/Y^{-n}\mathcal{N}_J)_0 = N/J^nN\).

Consequently, for all \(m, n \in \mathbb{N}\),

\[
\left( (\mathcal{F} \otimes_R \mathcal{G}) \otimes_R \frac{R}{(X-mY-n)} \right)_{(0,0)}
\equiv \left( \mathcal{F} \otimes R \frac{R}{(X-m)} \right)_0 \otimes_R \left( \mathcal{G} \otimes R \frac{R}{(Y-n)} \right)_0
= (an \ R\text{-resolution of } M/I^mM) \otimes_R (an \ R\text{-resolution of } N/J^nN).
\]
Proof (continued)

Therefore, we see

$$
H_c \left( (\mathcal{F} \otimes_R \mathcal{G}) \otimes_R \frac{R}{(X-mY-n)} \right)_{(0,0)}
$$

\[ \Rightarrow H_c \left( (\mathcal{F} \otimes_R \mathcal{G}) \otimes_R \frac{R}{(X-mY-n)} \right)_{(0,0)} \]

\[ \Rightarrow H_c \left( \mathcal{F} \otimes \mathcal{G} \otimes \frac{R}{(X-m)} \right)_0 \otimes_R \left( \mathcal{G} \otimes \mathcal{J} \otimes \frac{R}{(Y-n)} \right)_0 \]

\[ \Rightarrow \text{Tor}^R_c(M/I^mM, N/J^nN) \]
Proof (continued)

In summary, we have

\[
\left\{ H_c \left( (F \otimes_R G) \otimes_R \frac{R}{X-mY-n} \right) \right\}_{(m,n) \in \mathbb{N}^2} \text{ has linear growth over } R.
\]

\[
\left( H_c \left( (F \otimes_R G) \otimes_R \frac{R}{X-mY-n} \right) \right)_{(0,0)} \cong \text{Tor}^R_c(M/I^m M, N/J^n N).
\]

Finally, as primary decompositions behave well under scalar restriction and under submodule restriction, we see the family (of \( R \)-modules)

\[
\left\{ \text{Tor}^R_c(M/I^m M, N/J^n N) \right\}_{(m,n) \in \mathbb{Z}^2}
\]

has linear growth over \( R \).
Thank you