

# Tiling tripartite graphs with 3-colorable graphs

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## Abstract

For a fixed integer  $h \geq 1$ , let  $G$  be a tripartite graph with  $N$  vertices in each vertex class,  $N$  divisible by  $6h$ , such that every vertex is adjacent to at least  $2N/3 + h - 1$  vertices in each of the other classes. We show that if  $N$  is sufficiently large, then  $G$  can be tiled perfectly by copies of  $K_{h,h,h}$ . This extends the work in [19] and also gives a sufficient condition for tiling by any (fixed) 3-colorable graph. Furthermore, we show that this minimum-degree condition is best possible and provide very tight bounds when  $N$  is divisible by  $h$  but not by  $6h$ .

## 1 Introduction

Let  $H$  be a graph on  $h$  vertices, and let  $G$  be a graph on  $n$  vertices. Tiling problems in extremal graph theory are investigations of the condition or conditions under which  $G$  must contain many vertex disjoint copies of  $H$  (as subgraphs). An  $H$ -tiling of  $G$  is a subgraph of  $G$  which consists of vertex-disjoint copies of  $H$ . A *perfect  $H$ -tiling* of  $G$  is an  $H$ -tiling consisting of  $\lfloor n/h \rfloor$  copies of  $H$ . For clarity and consistency with other results in this area, we call a perfect  $H$ -tiling an  *$H$ -factor*. A very early tiling result is Dirac's theorem on Hamilton cycles [5], which implies that every  $n$ -vertex graph  $G$  with minimum degree  $\delta(G) \geq n/2$  contains a perfect matching (usually called 1-factor, instead of  $K_2$ -factor). Later Corrádi and Hajnal [4] studied the minimum degree of  $G$  that guarantees a  $K_3$ -factor. Hajnal and Szemerédi [8] settled the tiling problem for any complete graph  $K_h$  by showing that every  $n$ -vertex graph  $G$  with  $\delta(G) \geq (h-1)n/h$  contains a  $K_r$ -factor (it is easy to see that this is sharp). Using the celebrated Regularity Lemma of Szemerédi [23], Alon

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and Yuster [1, 2] obtained results on  $H$ -tiling for arbitrary  $H$ . Their results were later improved by various researchers [15, 12, 21, 18].

In this paper, we consider multipartite tiling, which restricts  $G$  to be an  $r$ -partite graph. When  $r = 2$ , The König-Hall Theorem (e.g. see [3]) answers the 1-factor problem for bipartite graphs. Wang [24, 25] considered  $K_{s,s}$ -factors in bipartite graphs for all  $s > 1$ , the second author [26] gave the best possible minimum degree condition for this problem.

In a tripartite graph  $G = (A, B, C; E)$ , the graphs induced by  $(A, B)$ ,  $(A, C)$  and  $(B, C)$  are called the **natural bipartite subgraphs** of  $G$ . Let  $\mathcal{G}_r(N)$  denote the family of  $r$ -partite graphs with  $N$  vertices in each of its partition sets. In an  $r$ -partite graph  $G$ ,  $\bar{\delta}(G)$  stands for the minimum degree from a vertex in one partition set to any other partition set. Fischer [7] gives almost perfect  $K_3$ -tilings in  $\mathcal{G}_3(N)$  with  $\bar{\delta}(G) \geq 2N/3$  and Johansson [9] gives a  $K_3$ -factor with a less stringent degree condition  $\bar{\delta}(G) \geq 2N/3 + O(\sqrt{N})$ . For all  $r > 2$ , Fischer [7] conjectured the following variant of Hajnal-Szemerédi Theorem.

**Conjecture 1.1 (Fischer [7])** *If  $G \in \mathcal{G}_r(N)$  satisfies  $\bar{\delta}(G) \geq \frac{r-1}{r}N$ , then  $G$  contains a  $K_r$ -factor.*

Recently, Szemerédi and the first author [20] proved Conjecture 1.1 for  $r = 4$ . However, Conjecture 1.1 is false when  $r = 3$ : the following construction of Magyar and the first author, [19], provides a counterexample. Let  $\Gamma_r \in \mathcal{G}_r(r)$  have vertices  $\{h_{i,j} : i = 1, \dots, r; j = 1, \dots, r\}$  and the adjacency rule as follows:  $h_{i,j} \sim h_{i',j'}$  if  $i \neq i'$  and  $j \neq j'$  and either  $j$  or  $j'$  is in  $\{1, \dots, r-2\}$ . Also,  $h_{i,r-1} \sim h_{i',r-1}$  and  $h_{i,r} \sim h_{i',r}$  for  $i \neq i'$ . No other edges exist. It is easy to check that  $\bar{\delta}(\Gamma_r) = r-1$  and when  $r$  is odd,  $\Gamma_r$  contains no  $K_r$ -factor.

Nevertheless, [19] showed that, if  $N$  is an odd multiple of 3, the so-called blow-up graph  $\Gamma_3(N) \in \mathcal{G}_3(N)$  (where each edge of  $\Gamma_3$  is replaced with a  $K_{N/3, N/3}$  and each non-edge is replaced by an  $(N/3) \times (N/3)$  bipartite graph with no edges) is the unique exception for Conjecture 1.1 in the case  $r = 3$ . As a result, this gives the following Corrádi-Hajnal-type theorem.

**Theorem 1.2 (Magyar-M. [19])** *If  $G \in \mathcal{G}_3(N)$  satisfies  $\bar{\delta}(G) \geq (2/3)N + 1$ , then  $G$  contains a  $K_3$ -factor.*

In this paper we extend this result to all 3-colorable graphs. Our main result is the following theorem.

**Theorem 1.3** *Fix a positive integer  $h$ . Let  $f(h)$  be the smallest value for which there exists an  $N_0$  such that if  $G$  is a balanced tripartite graph on  $3N$  vertices,  $N \geq N_0$ ,  $h \mid N$ , and each vertex is adjacent to at least  $f(h)$  vertices in each of the other classes, then  $G$  contains a  $K_{h,h,h}$ -factor.*

If  $N = (6q + r)h$  with  $0 \leq r < 6$ , then

$$\begin{aligned} f(h) &= \frac{2N}{3} + h - 1, & \text{if } r = 0; \\ h \lceil \frac{2N}{3h} \rceil + h - 2 &\leq f(h) \leq h \lceil \frac{2N}{3h} \rceil + h - 1, & \text{if } r = 1, 2, 4, 5; \\ \frac{2N}{3} + h - 1 &\leq f(h) \leq \frac{2N}{3} + 2h - 1, & \text{if } r = 3. \end{aligned}$$

So, the result is tight for  $N = 6h$ , almost tight unless  $N$  is an odd multiple of 6 and, in the worst case, the upper and lower bounds differ by  $h$ .

Clearly the complete tripartite graph  $K_{h,h,h}$  can be perfectly tiled by any 3-colorable graph on  $h$  vertices. Since  $f(h) \leq \frac{2N}{3} + 2h - 1$  whenever  $N$  is divisible by  $h$ , we have the following corollary.

**Corollary 1.4** *Let  $H$  be a 3-colorable graph of order  $h$ . There exists a positive integer  $N_0$  such that if  $N \geq N_0$  and  $N$  divisible by  $h$ , then every  $G \in \mathcal{G}_3(N)$  with  $\bar{\delta}(G) \geq \frac{2N}{3} + 2h - 1$  contains an  $H$ -factor.*

It is well known that every graph  $G$  on  $n = Nr$  vertices contains a subgraph  $G' \in \mathcal{G}_r(N)$  with  $\bar{\delta}(G) \geq \delta(G)/r - o(n)$  (following from a random balanced partition of the vertices of  $G$ ). Consequently Corollary 1.4 gives another proof of the Alon-Yuster theorem [2] for 3-colorable graphs as follows: Fix a 3-colorable graph  $H$  of order  $h$  and let  $G$  be a graph of order  $n = 3N$  such that  $N$  is sufficiently large and divisible by  $h$ . If  $\delta(G) \geq 2n/3 + \varepsilon n$  for some  $\varepsilon > 0$ , then we first find a subgraph  $G' \in \mathcal{G}_3(N)$  with  $\bar{\delta}(G) \geq (2/3)N + 2h - 1$ , and then apply Corollary 1.4 to  $G'$  obtaining an  $H$ -factor in  $G'$ , hence in  $G$ .

The proof of Theorem 1.3 naturally falls into two parts as those of other tiling results [15, 21, 19, 20]. In the first stage, we prove a result that resembles the stability theorem of Simonovits [22]; namely, any balanced tripartite graph with a slightly weaker degree condition either contains an  $K_{h,h,h}$ -factor, or is in a class of extremal graphs. In the second stage, we show that any graph close to the extremal graphs contains an  $K_{h,h,h}$ -factor. This approach seems to be a useful tool for obtaining exact results on graphs or hypergraphs [13, 10, 11, 19, 20]. Our second stage turns out to be lengthy and intricate due to the fact that we must ensure that, when sets are partitioned, they must be divisible by  $h$ .

The structure of the paper is as follows. After stating the Regularity Lemma and Blow-up Lemma in Section 2.1, we prove the so-called ‘‘fuzzy’’ case (Theorem 3.1) in Section 3 and the extreme case (Theorem 4.2) in Section 4. The graphs that provide the lower bounds for  $f(h)$  in Theorem 1.3 are constructed in Section 5.

## 2 Tools and Definitions

### 2.1 The Regularity Lemma and Blow-up Lemma

The Regularity Lemma and the Blow-up Lemma are main tools in the proof of the so-called “fuzzy” case. Let “+” designate a disjoint union of sets. We define the usual concepts of  $\epsilon$ -regularity and  $(\epsilon, \delta)$ -super-regularity and state the version of the Regularity Lemma that we use. See [16] and [17]. In this paper, when floors and ceilings are not crucial and do not effect the result, we ignore them.

**Definition 2.1** *The bipartite graph  $G = (A, B, E)$  is  $\epsilon$ -regular if*

$$X \subset A, \quad Y \subset B, \quad |X| > \epsilon|A|, \quad |Y| > \epsilon|B|$$

*imply  $|d(X, Y) - d(A, B)| < \epsilon$ , otherwise we say  $G$  is  $\epsilon$ -irregular.*

**Definition 2.2**  *$G = (A, B, E)$  is  $(\epsilon, \delta)$ -super-regular if*

$$X \subset A, \quad Y \subset B, \quad |X| > \epsilon|A|, \quad |Y| > \epsilon|B|$$

*imply  $d(X, Y) > \delta$  and*

$$\deg(a) > \delta|B|, \quad \forall a \in A \quad \text{and} \quad \deg(b) > \delta|A|, \quad \forall b \in B.$$

**Lemma 2.3 (Regularity Lemma - Degree Form)** *For every positive  $\epsilon$  there is an  $M = M(\epsilon)$  such that if  $G = (V, E)$  is any graph and  $d \in [0, 1]$  is any real number, then there is a partition of the vertex set  $V$  into  $\ell + 1$  clusters  $V_0, V_1, \dots, V_\ell$  and there is a subgraph  $G' = (V, E')$  with the following properties:*

- $\ell \leq M$ ,
- $|V_0| \leq \epsilon|V|$ ,
- all clusters  $V_i, i \geq 1$ , are of the same size  $L \leq \epsilon|V|$ ,
- $\deg_{G'}(v) > \deg_G(v) - (d + \epsilon)|V|, \forall v \in V$ ,
- $G'|_{V_i} = \emptyset$  ( $V_i$  are independent in  $G'$ ),
- all pairs  $G'|_{V_i \times V_j}, 1 \leq i < j \leq \ell$ , are  $\epsilon$ -regular, each with density either 0 or exceeding  $d$ .

The proof of the regularity lemma (see [23]) begins with any equipartition of the vertex set and refines it into a Szemerédi partition, as defined above. So, when we apply Lemma 2.3 to a balanced tripartite graph on  $3N$  vertices with  $N$  large enough, we can ensure that each cluster, other than  $V_0$ , is a subset of exactly one piece of the tripartition.

We will also need the Blow-up Lemma of Komlós, Sárközy and Szemerédi. The graph  $H$  can be embedded into graph  $G$  if  $G$  contains a subgraph isomorphic to  $H$ .

**Lemma 2.4 (Blow-up Lemma [14])** *Given a graph  $R$  of order  $r$  and positive parameters  $\delta, \Delta$ , there exists an  $\epsilon > 0$  such that the following holds: Let  $N$  be an arbitrary positive integer, and let us replace the vertices of  $R$  with pairwise disjoint  $N$ -sets  $V_1, V_2, \dots, V_r$  (blowing up). We construct two graphs on the same vertex-set  $V = \cup V_i$ . The graph  $R(N)$  is obtained by replacing all edges of  $R$  with copies of the complete bipartite graph  $K_{N,N}$  and a sparser graph  $G$  is constructed by replacing the edges of  $R$  with some  $(\epsilon, \delta)$ -super-regular pairs. If a graph  $H$  with maximum degree  $\Delta(H) \leq \Delta$  can be embedded into  $R(N)$ , then it can be embedded into  $G$ .*

### 3 The Fuzzy Tripartite Theorem

The purpose of this section is to prove the following so-called fuzzy tripartite theorem. We say that  $G$  is in *the extreme case with parameter  $\Delta$*  if  $G$  has three sets of size  $\lfloor N/3 \rfloor$ , each in a different vertex class, with pairwise density at most  $\Delta$ . Recall that  $\mathcal{G}_3(N)$  is the family of balanced tripartite graphs with three parts, each of size  $N$ .

**Theorem 3.1** *Given any positive integer  $h$  and a  $\Delta > 0$ , sufficiently small, there exists an  $\epsilon > 0$  and an integer  $N_0 = N_0(\Delta, h)$  such that whenever  $N \geq N_0$ , and  $h$  divides  $N$ , the following occurs: If  $G = (V^{(1)}, V^{(2)}, V^{(3)}; E) \in \mathcal{G}_3(N)$  such that  $\bar{\delta}(G) \geq (2/3 - \epsilon)N$ , then either  $G$  has a subgraph which consists of  $N/h$  vertex-disjoint copies of the complete tripartite graph  $K_{h,h,h}$  or  $G$  is in the extreme case with parameter  $\Delta$ .*

#### Proof of Theorem 3.1

As usual, there is a sequence of constants:

$$\epsilon \ll \epsilon_1 \ll \delta \ll d_1 \ll \Delta_0 \ll \Delta \ll h^{-1}.$$

Begin with a tripartite graph  $G = (V^{(1)}, V^{(2)}, V^{(3)}; E)$  with  $|V^{(1)}| = |V^{(2)}| = |V^{(3)}| = N$  in which each vertex is adjacent to at least  $(2/3 - \epsilon)N$  vertices in each of the other classes. Apply the Regularity Lemma (Lemma 2.3) with  $\epsilon_1$  and  $d_1$ , partitioning each  $V_i$  into  $\ell$  clusters  $V_1^{(i)} + \dots + V_\ell^{(i)}$

of size  $L \leq 3\epsilon_1 N$  and an exceptional set  $V_0^{(i)}$  of size at most  $3\epsilon_1 N$ . Let us define  $G_r$  to be the reduced graph defined as usual. It is important to observe that in the proof, the exceptional sets will increase in size, but will always remain of size  $O(\epsilon_1 N)$ .

It is a routine calculation to see that the reduced graph  $G_r$  (defined in the usual way where clusters are adjacent if the pair is  $\epsilon_1$ -regular of density at least  $d_1$ ) has the condition that every cluster is adjacent to at least  $(2/3 - d_1)\ell$  clusters in each of the other vertex classes.

### Step 1: Finding a triangle cover in $G_r$

Here we can apply Lemma 3.2, the Almost-covering Lemma, (Lemma 2.2 in [19]) repeatedly to  $G_r$  to get a decomposition of  $G_r$  into cluster-disjoint copies of  $K_3$  and at most 9 clusters. If this is not possible, then  $G$  must be in the extreme case.

**Lemma 3.2 (Almost-covering Lemma [19])** *Let  $H$  be a balanced tripartite graph on  $3M$  vertices such that each vertex is adjacent to at least  $(2/3 - \epsilon)M$  vertices in each of the other classes. Let  $\mathcal{T}_0$  be a partial  $K_3$ -tiling in  $H$  with  $|\mathcal{T}| < M - 3$ . Then, either*

1. *there exists a partial  $K_3$ -tiling  $\mathcal{T}'$  with  $|\mathcal{T}'| > |\mathcal{T}|$  but  $|\mathcal{T}' \setminus \mathcal{T}| \leq 15$ , or*
2.  *$H$  has 3 subsets in 3 vertex classes of size  $\lfloor M/3 \rfloor$  with pairwise density at most  $\Delta_0$ .*

Note that the second case of Lemma 3.2 implies that  $G_r$  is in the extreme case with parameter  $\Delta_0$  and so  $G$  itself is in the extreme case with parameter  $\Delta \gg \Delta_0$ . The fact that  $|\mathcal{T}' \setminus \mathcal{T}| \leq 15$  is not important here but is crucial to arguments in Step 4.

We put the vertices from the clusters that are outside of the  $K_3$ -factor into the corresponding exceptional set. For simplicity of notation, we still denote the remaining graph by  $G_r$  and assume that each vertex class of  $G_r$  has size  $\ell$ . The cluster-triangles which cover  $G_r$  are called  $S_1, S_2, \dots, S_\ell$ , where  $S_i = \{S_i^{(1)}, S_i^{(2)}, S_i^{(3)}\}$  with  $S_i^{(j)} \subseteq V^{(j)}$  for  $j = 1, 2, 3$ .

### Step 2: Making pairs in $S_i$ super-regular

For each cluster-triangle,  $S_i$ , remove some vertices from it to make each pair not just regular, but super-regular. That is, remove a vertex  $v$  from a cluster in  $S_i$  and place it in the exceptional set if  $v$  has fewer than  $(d_1 - \epsilon_1)L$  neighbors in each of the other clusters of  $S_i$ . By  $\epsilon_1$ -regularity, there are at most  $2\epsilon_1 L$  such vertices in each cluster. Remove more vertices if necessary to ensure that each non-exceptional cluster is of the same size, which is at least  $(1 - 2\epsilon_1)L$  and divisible by  $h$ .

The Slicing Lemma is important for verifying that regularity is maintained when small modifications are made to the clusters:

**Lemma 3.3 (Slicing Lemma, Fact 1.5 in [19])** *Let  $(A, B)$  be an  $\varepsilon$ -regular pair with density  $d$ , and, for some  $\alpha > \varepsilon$ , let  $A' \subset A$ ,  $|A'| \geq \alpha|A|$ ,  $B' \subset B$ ,  $|B'| \geq \alpha|B|$ . Then  $(A', B')$  is an  $\varepsilon'$ -regular pair with  $\varepsilon' = \max\{\varepsilon/\alpha, 2\varepsilon\}$ , and for its density  $d'$ , we have  $|d' - d| < \varepsilon$ .*

Summarizing, any pair of clusters which was  $\varepsilon_1$ -regular with density at least  $d_1$  is now  $(2\varepsilon_1)$ -regular with density at least  $d_1 - \varepsilon_1$ , as long as  $\varepsilon_1 < 1/4$ . Furthermore, each pair in a cluster-triangle  $S_i$  is  $(2\varepsilon_1, d_1 - 3\varepsilon_1)$ -super-regular. Each of the three exceptional sets are now of size at most  $\varepsilon_1 N + \ell(2\varepsilon_1 L) \leq 3\varepsilon_1 N$ . The other clusters have the same number of vertices, which is at least  $(1 - 2\varepsilon_1)L$  and is divisible by  $h$ .

**Remark:** Because each triple  $(S_i^{(1)}, S_i^{(2)}, S_i^{(3)})$ , is super-regular, we can apply the Blow-up Lemma to them (once we modify them to be of equal size, divisible by  $h$ ) so that they contain a spanning subgraph of vertex-disjoint copies of  $K_{h,h,h}$ .

### Step 3: Create auxiliary triangles

In this step we link each cluster to the corresponding one in the first cluster-triangle,  $S_1$ . Its purpose is handling the last constant number of leftover vertices in Step 5.

**Definition 3.4** *In a tripartite graph  $G = (V^{(1)}, V^{(2)}, V^{(3)}; E)$ , one vertex  $x \in V^{(1)}$  (the cases of  $x \in V^{(2)}$  or  $V^{(3)}$  are defined similarly) is **reachable** from another vertex  $y \in V^{(1)}$ , if there is a chain of triangles  $T_1, \dots, T_{2k}$  with  $T_j = \{T_j^{(1)}, T_j^{(2)}, T_j^{(3)}\}$  and  $T_j^{(i)} \in V^{(i)}$  for  $i = 1, 2, 3$  such that the following occurs:*

1.  $x = T_1^{(1)}$  and  $y = T_{2k}^{(1)}$ ,
2.  $T_{2j-1}^{(2)} = T_{2j}^{(2)}$  and  $T_{2j-1}^{(3)} = T_{2j}^{(3)}$ , for  $j = 1, \dots, k$ , and
3.  $T_{2j}^{(1)} = T_{2j+1}^{(1)}$ , for  $j = 1, \dots, k-1$ .

The Reachability Lemma (Lemma 2.6 in [19]) states that, in the reduced graph  $G_r$ , one cluster is reachable from any other cluster in the same class using at most *four* cluster-triangles, unless  $G$  is in the extreme case. Thus, any cluster of  $S_1$  is reachable from every other cluster in the same class using at most 4 cluster-triangles (whose clusters come from at most 6 different  $S_i$ ). Figure 1 illustrates how  $S_1^{(1)}$  is reachable from cluster  $C$ .

Let  $C$  be a cluster in  $V^{(1)}$  and let  $T_1, T_2$  or  $T_1, T_2, T_3, T_4$  be cluster-triangles that witness the fact that  $C$  is reachable from  $S_1^{(1)}$ . Note that  $T_1 \cap V^{(1)} = S_1^{(1)}$  and either both  $k = 1$  and  $T_2 \cap V^{(1)} = C$  or  $k = 2$ ,  $T_2 \cap V^{(1)} = C'$  and  $T_4 \cap V^{(1)} = C$ .

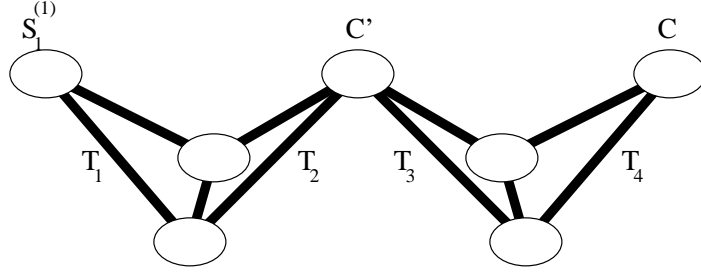


Figure 1: An illustration of how cluster  $S_1^{(1)}$  is reachable from a cluster  $C$ .

If  $k = 1$ , then create a copy of  $K_{h,h,h}$ , called  $H'$ , in the cluster triangle  $T_1$ , as well as an extra vertex in  $C$  adjacent to the vertices in  $H' \cap V^{(2)}$  and  $H' \cap V^{(3)}$ . This is possible because all involved pairs are regular of nontrivial density and  $h$  is a constant compared with  $L$ , the size of clusters.

If  $k = 2$ , then create two copies of  $K_{h,h,h}$ . The first is again called  $H'$ , in the cluster triangle  $T_1$ , as well as an extra vertex in the  $V^{(1)}$  cluster that forms  $T_2 \cap T_3$ , which is adjacent to the vertices in  $H' \cap V^{(2)}$  and  $H' \cap V^{(3)}$ . The second copy of  $K_{h,h,h}$ , called  $H''$ , is in the cluster triangle  $T_3$  and there is a single vertex in  $C$ , which is adjacent to the vertices in  $H'' \cap V^{(2)}$  and  $H'' \cap V^{(3)}$ .

Color all of the vertices in  $H'$  and in  $H''$  (if it exists) *red* and the additional vertex in  $C$  and in  $C'$  (if it exists) *orange*. If a vertex is not colored, we will heretofore call it *uncolored*. Repeat this  $6h$  times for every cluster  $C$  in  $G_r$ , ensuring that all such red copies of  $K_{h,h,h}$  and orange vertices are vertex-disjoint.

This process of creating red copies of  $K_{h,h,h}$  may result in a discrepancy of uncolored vertices in the three clusters of some  $S_j$ 's. A cluster may have at most  $(3\ell)(6h)(h) = 18\ell h^2$  red vertices because there are  $3\ell$  clusters  $C$ , the process is iterated  $6h$  times for each  $C$  and no cluster gets more than  $h$  vertices colored red with each iteration. Similarly, no cluster gets more than  $\ell + 1$  orange vertices. We will remove some uncolored vertices from each cluster, placing them in an exceptional set. This will be done to ensure that the number of uncolored vertices in each cluster is the same and is divisible by  $h$ . Thus, at most  $18\ell h^2 + (\ell + 1) + (h - 1)$  vertices are removed from any cluster. The sizes of exceptional sets are, thus, increased by at most  $(18\ell h^2 + \ell + h)\ell \leq 20\ell^2 h^2$ , a constant.

Summarizing, if we have  $20\ell^2 h^2 \leq \varepsilon_1 L$ , then any pair which, originally, was  $\varepsilon_1$ -regular with density at least  $d_1$  has that its uncolored vertices form a pair which is  $(4\varepsilon_1)$ -regular with density at least  $d_1 - 2\varepsilon_1$ , as long as  $\varepsilon_1 < 1/4$ . Furthermore, the uncolored vertices in each pair of some cluster-triangle  $S_i$  is  $(2\varepsilon_1, d_1 - 4\varepsilon_1)$ -super-regular. Each of the three exceptional sets are now of size at most  $3\varepsilon_1 N + 20\ell^2 h^2 \leq 4\varepsilon_1 N$ . The other (non-exceptional) clusters have at least  $(1 - 2\varepsilon_1)L$  vertices with at most  $\varepsilon_1 L$  red vertices. In each non-exceptional cluster, the number of orange vertices is at most  $\varepsilon_1 L$ . The number of uncolored vertices is at least  $(1 - 4\varepsilon_1)L$  and is divisible by  $h$ .

**Remark:** This preprocessing ensures that we may later transfer at most  $6h$  vertices from any cluster  $C$  to  $S_1$  in the following sense: Without loss of generality, suppose  $C$  is a cluster in  $V^{(1)}$ . In the case where  $k = 2$ , identify an orange vertex in  $C$  and its corresponding red subgraphs  $H'$  and  $H''$  and the orange vertex in the cluster  $C'$ . (The case where  $k = 1$  is similar but simpler.)

Recolor the orange vertex in  $C$  to be red. Make a vertex from the set  $H'' \cap C'$  to be uncolored and recolor the corresponding orange vertex in  $C'$  to be red. Finally, uncolor a vertex from the set  $H' \cap S_1^{(1)}$ . Except for  $C$  and  $S_1^{(1)}$ , the number of uncolored plus orange vertices remains the same in every cluster. This number is decreased by 1 in  $C$  but is increased by 1 in  $S_1^{(1)}$ . We will do this in Step 5.

#### Step 4: Reducing the sizes of exceptional sets to $6h$

Consider the exceptional sets  $V_0^{(i)}$  for  $i = 1, 2, 3$ , which are all of the same size, at most  $4\epsilon_1 N$ . We will show that we can make them of size less than  $6h$ . So, suppose  $|V_0^{(1)}| \geq 6h$ .

We will find  $4h$  vertices in each exceptional set, bundling them into 4 sets of size  $h$ . In the algorithm below, we will place at least one bundle from each vertex class into  $h$  vertex-disjoint copies of  $K_{h,h,h}$  and, together with at most 15 additional copies of  $K_{h,h,h}$ , we can remove them from the graph, reducing the number of vertices in the exceptional set by at least  $h$  vertices.

First, we observe that each vertex in the exceptional sets can be regarded as a vertex in some non-exceptional cluster in the following sense: Given an  $S_j = (S_j^{(1)}, S_j^{(2)}, S_j^{(3)})$  and a vertex  $v \in V_0^{(1)}$ . If  $v$  is adjacent to at least  $\delta L$  uncolored vertices in  $S_j^{(2)}$  and  $S_j^{(3)}$ , then we may remove a copy of  $K_{h,h,h}$  which consists of  $v$ ,  $h - 1$  vertices from  $S_j^{(1)}$  and  $h$  vertices from each of  $S_j^{(2)}$  and  $S_j^{(3)}$ . This is easy to do because each pair is regular with nontrivial density.

Accordingly, we say a vertex  $v \in V^{(i)}$  belongs to a cluster  $S_j^{(i)}$ , for some,  $i$  if  $v$  is adjacent to at least  $\delta L$  uncolored vertices of each of the other clusters in  $S_j$ . Using the minimum-degree condition, the number of clusters in some other vertex class for which  $v$  is adjacent to fewer than  $\delta L$  uncolored vertices is at most

$$\frac{(1/3 + \epsilon)N}{(1 - 3\epsilon_1)L - \delta L} \leq \frac{(1/3 + \epsilon)\ell}{(1 - 3\epsilon_1 - \delta)(1 - \epsilon_1)}. \quad (1)$$

As long as  $\delta$  is small enough, the expression in (1) is at most  $(1/3 + \delta)\ell$ . Thus,  $v$  is adjacent to at least  $\delta L$  uncolored vertices in at least  $(2/3 - \delta)\ell$  clusters in  $V^{(k)}$ ,  $k \neq i$ . Hence, each vertex in  $V_0^{(i)}$  belongs to at least  $(1/3 - 2\delta)\ell$  clusters.

Therefore, as long as  $|V_0^{(1)}| \geq 3h$  and  $\delta$  is small enough ( $\delta < 1/(6h)$  is enough), the Pigeonhole Principle guarantees that there is a cluster  $C$  and set of  $h$  vertices in  $V_0^{(1)}$  such that each these  $h$  vertices belongs to  $C$ . Since  $|V_0^{(1)}| \geq 6h$ , we can repeat this four times in order to find four disjoint

$h$ -element subsets  $W_1^{(1)}, \dots, W_4^{(1)}$  of  $V_0^{(1)}$  whose vertices belong to disjoint clusters  $C_1^{(1)}, \dots, C_4^{(1)}$ , respectively.

Next we will show that, for  $i = 1, 2, 3$ , there is some  $j \in \{1, 2, 3, 4\}$  for which one of these sets  $W_j^{(i)}$  of  $h$  vertices can be *inserted* into  $C_j^{(i)}$ .

To see how this insertion works, it is useful to construct an auxiliary graph  $\tilde{G}$ . The vertex set  $V(\tilde{G})$  consists of  $\ell + 4$  vertices in each of three vertex classes. The first  $\ell$  vertices in each class represent the first  $\ell$  clusters in each class and two are adjacent if, originally, the pair of clusters was  $\varepsilon_1$  regular with density at least  $d_1$ . The remaining 4 vertices in each class are merely duplicates of the vertices representing the special clusters  $C_1^{(i)}, \dots, C_4^{(i)}$  for  $i = 1, 2, 3$ . Denote the duplicate of  $C_j^{(i)}$  by  $\tilde{C}_j^{(i)}$ . They are adjacent to the same vertices as their originals, but there are no edges between any of these 12 duplicates.

Let  $\mathcal{T}$  be the partial triangle-cover of  $\tilde{G}$  corresponding to the triangle-cover  $S_1, \dots, S_\ell$  and apply the Almost-covering Lemma (Lemma 3.2) to  $\tilde{G}$  with this  $\mathcal{T}$ . The lemma provides a larger partial triangle-cover  $\mathcal{T}'$  which differs from  $\mathcal{T}$  by at most 15 triangles. We now create vertex-disjoint copies of  $K_{h,h,h}$  as follows: For each triangle in  $\mathcal{T}' \setminus \mathcal{T}$ , find a copy of  $K_{h,h,h}$  in the uncolored vertices of the triple represented by that triangle.

To see how to deal with the duplicate clusters, suppose  $\tilde{C}_1^{(1)}$  is a vertex in  $\mathcal{T}' \setminus \mathcal{T}$ . For each of the  $h$  vertices that belong to  $\tilde{C}_1^{(1)}$ , place it in a  $K_{h,h,h}$  which contains  $h - 1$  vertices in  $\tilde{C}_1^{(1)}$  and  $h$  in each of the other clusters of the  $S_i$  which contains  $\tilde{C}_1^{(1)}$ . All of these copies of  $K_{h,h,h}$  can be removed from the graph entirely, they will be a part of the final  $K_{h,h,h}$ -factor of  $G$ . In the process of creating  $\mathcal{T}'$ , there may be a cluster that was covered by  $\mathcal{T}$  but is not covered by the larger  $\mathcal{T}'$ . In such a case, take an arbitrary set of  $h$  uncolored vertices from that cluster and place it into the leftover set. Since  $|\mathcal{T}'| > |\mathcal{T}|$ , the net change in each leftover set is the same and they each lose at least  $h$  vertices. Regardless, no cluster loses more than  $h^2 + h$  vertices.

We repeat this process until the number of vertices remaining in each exceptional set is at most  $6h$ . There is one caveat: If too many vertices are removed from the clusters of  $S_i$ , then we will not be able to apply the Blow-up Lemma later. So, if in the process of executing this algorithm, at least  $(\delta/2)L$  vertices are used from a cluster of  $S_i$ , then we say that  $S_i$  is *dead*.

The number of dead cluster-triangles is not very large. To see this, there are three ways for vertices to leave a cluster. First, they are placed in a  $K_{h,h,h}$  with a vertex from the leftover set, so each vertex class  $V^{(i)}$  loses at most  $3|V_0^{(i)}|h$  vertices in this way. Second, each time  $h$  vertices are inserted, there are at most 15 vertices that are a vertex in  $\mathcal{T}' \setminus \mathcal{T}$  and so could lose a total of  $15h$  vertices to a copy of  $K_{h,h,h}$ . Third, there are at most 3 that are vertices uncovered by  $\mathcal{T}'$  and so could lose  $3h$  vertices have to be placed from a cluster into the . Since this algorithm is executed  $|V_0^{(i)}|/h$  times,

the total number of vertices that leave clusters is at most

$$3|V_0^{(i)}|h + \left(|V_0^{(i)}|/h\right)(15h + 3h) = |V_0^{(i)}|(3h + 18) \leq 6\epsilon_1 N(3h + 18).$$

The number of dead cluster triangles is at most

$$\frac{6\epsilon_1 N(3h + 18)}{(\delta/2)L} = \frac{36(h + 6)\epsilon_1}{\delta(1 - \epsilon_1)}\ell.$$

So, as long as  $\epsilon_1 \ll \delta \ll d_1$ , the number of dead clusters is at most  $d_1\ell$  and the Almost-covering Lemma (Lemma 3.2) can be applied to the live clusters without changing the result because each cluster is adjacent to at least  $(2/3 - d_1)\ell - d_1\ell$  clusters.

Summarizing, the graph  $G$  consists of some vertex-disjoint copies of  $K_{h,h,h}$ . The remaining vertices induce a graph with clusters that form triangles  $S_1, \dots, S_\ell$ . In each cluster, the number of uncolored vertices is at least  $(1 - 4\epsilon_1)L - \delta L \geq (1 - d_1/2)L$  and is divisible by  $h$ . For  $i = 1, \dots, \ell$ , the uncolored vertices in  $S_i$  form a triple that is pairwise  $(8\epsilon_1, d_1 - \delta)$ -super-regular with density at least  $d_1 - 3\epsilon_1$ , as long as  $\delta \geq 4\epsilon_1$ . The edges between other pairs of clusters are no longer relevant.

### Step 5: Inserting the last $\leq 6h$ leftover vertices

Assume that each exceptional set has at most  $6h$  vertices. Consider a vertex  $x \in V_0^{(1)}$  and suppose  $x$  belongs to some cluster  $C$ . In the remark at the end of Step 2 we discussed how to move a vertex from  $C$  to  $S_1^{(1)}$ . So, we place  $x$  in  $C$ , uncolor one of the orange vertices in  $C$  and proceed by using the red vertices and orange in the manner prescribed in Step 3, until  $S_1^{(1)}$  has an extra uncolored vertex. Repeat until all of the leftover vertices have been assigned to a cluster.

Then, uncolor all remaining orange vertices and remove the red copies of  $K_{h,h,h}$ . It remains to show that the vertices that remain in each triple  $S_i$  themselves form a  $K_{h,h,h}$ -factor. The number of uncolored vertices has not decreased but has increased by at most  $\epsilon_1 L$ . Now, we just have to establish that uncolored vertices in the pairs of a cluster-triangle remain super-regular for some set of parameters.

Each vertex in the leftover set is adjacent to at least  $(\delta/2)L$  vertices in the other live clusters if it belongs to  $C$  because it had been adjacent to  $\delta L$  before Step 4 and at most  $(\delta/2)L$  vertices are removed by Step 4. Some straightforward calculations, which we neglect to include here, show that an  $(8\epsilon_1)$ -regular pair, with each cluster of size at least  $(1 - d_1/2)L$ , will be  $(2\sqrt{\epsilon_1})$ -regular if at most  $\epsilon_1 L$  vertices are added to each set, as long as  $\epsilon_1 \ll d_1 \ll h^{-1}$ .

Therefore, the pairs of vertices in each  $S_i$  are  $(2\sqrt{\epsilon_1}, \delta/2)$ -super-regular and we can apply the Blow-up Lemma to each  $S_j$  to complete the  $K_{h,h,h}$ -factor of  $G$ .

## 4 The Extremal Case

Before we deal with the extremal case, we make the solution precise by describing a specific exclusionary case, which we deal with in Section 4.5.

**Definition 4.1** *A balanced tripartite graph  $G$  on  $3N$  vertices is in the very extreme case if the following occurs: First, there are integers  $N, q$  such that  $N = (6q + 3)h$ . Second, there are sets  $A_j^{(i)} \subseteq V^{(i)}$  for  $i, j \in \{1, 2, 3\}$ , each with size at least  $2qh + 1$ , such that if  $v \in A_j^{(i)}$  then  $v$  is nonadjacent to at most  $3h - 3$  vertices in  $A_{j'}^{(i')}$  whenever the pair  $(A_j^{(i)}, A_{j'}^{(i')})$  corresponds to an edge in the graph  $\Gamma_3$  with respect to the usual correspondence.*

The Main Theorem is proven by verifying the following:

**Theorem 4.2** *Given any positive integer  $h$ , there exists a  $\Delta$ ,  $0 < \Delta \ll h^{-1}$  and  $N_0 = N_0(h)$  such that whenever  $N \geq N_0$  and  $h$  divides  $N$ , the following occurs: If  $G = (V^{(1)}, V^{(2)}, V^{(3)}; E)$  is a balanced tripartite graph on  $3N$  vertices and  $G$  is in the extreme case with parameter  $\Delta$  and  $\bar{\delta}(G) \geq h \lceil \frac{2N}{3h} \rceil + h - 1$ , then, either  $G$  has a  $K_{h,h,h}$ -factor or  $N$  is an odd multiple of  $3h$  and  $G$  is in the very extreme case.*

*If  $G$  is in the very extreme case, we can find the  $K_{h,h,h}$ -factor if  $\bar{\delta}(G) \geq h \lceil \frac{2N}{3h} \rceil + 2h - 1$ .*

Throughout all of Section 4, assume that  $G$  is minimal, *i.e.*, no edge of  $G$  can be deleted so that the minimum degree condition still holds. We will have the usual sequence of constants:

$$\Delta \ll \Delta_1 \ll \Delta_2 \ll \Delta_3 \ll \Delta_4 \ll \Delta_5 \ll h^{-1}.$$

We will assume for Parts 1, 2 and 3a (Sections 4.1, 4.2 and 4.3, respectively) that  $\bar{\delta}(G) \geq h \lceil \frac{2N}{3h} \rceil + h - 1$ . In Part 3b (Section 4.4), we will begin with the same assumption on  $\bar{\delta}$ , until we are left with the very extreme case. Then we will allow  $\bar{\delta}(G) \geq h \lceil \frac{2N}{3h} \rceil + 2h - 1$  in Section 4.5 to complete the proof.

**Definition 4.3** *For  $\delta$ ,  $0 < \delta < 1$  a graph  $H$  and positive integer  $M$ , we say a graph  $G$  is  $\delta$ -approximately  $H(M)$  if  $V(G)$  can be partitioned into  $|V(H)|$  nearly-equally sized pieces, each corresponding to a vertex of  $H$  so that for vertices  $v, w \in V(H)$  with  $v \not\sim_H w$ , the parts of  $V(G)$  corresponding to  $v$  and  $w$  have density between them less than  $\delta$ .*

### 4.1 Part 1: The basic extremal case

For Part 1, we will prove that either a  $K_{h,h,h}$ -factor exists in  $G$ , or  $G$  is in Part 2.

Let  $A^{(i)} \subset V^{(i)}$  for  $i = 1, 2, 3$  be the three pairwise sparse sets given by the statement of the theorem and  $B^{(i)} = V^{(i)} \setminus A^{(i)}$  for  $i = 1, 2, 3$ . We then define  $\tilde{A}^{(i)}$  to be the “typical” vertices with respect to  $A^{(i)}$ ,  $\tilde{B}^{(i)}$  to be “typical” with respect to  $B^{(i)}$ , and  $C^{(i)}$  are what remain. Formally, for  $i = 1, 2, 3$ ,

$$\begin{aligned}\tilde{A}^{(i)} &= \left\{x \in V^{(i)} : \deg_{A^{(j)}}(x) \leq \Delta_1 |A^{(j)}|, \forall j \neq i\right\} \\ \tilde{B}^{(i)} &= \left\{y \in V^{(i)} : \deg_{A^{(j)}}(y) \geq (1 - \Delta_1) |A^{(j)}|, \forall j \neq i\right\} \\ C^{(i)} &= V^{(i)} \setminus \left(A^{(i)} \cup B^{(i)}\right)\end{aligned}$$

As a result, we have that  $|B^{(i)} \setminus \tilde{B}^{(i)}| \leq (2\Delta/\Delta_1)|B^{(i)}|$  and  $|A^{(i)} \setminus \tilde{A}^{(i)}| \leq (2\Delta/\Delta_1)|A^{(i)}|$ . So, with  $\Delta_1 = \Delta^{1/3}$ ,  $|\tilde{B}^{(i)}| \geq (1 - 2\Delta_1^2)|B^{(i)}| \geq (1 - 2\Delta_1^2)(2N/3)$  and  $|\tilde{A}^{(i)}| \geq (1 - 2\Delta_1^2)|A^{(i)}| \geq (1 - 2\Delta_1^2)(N/3)$ . We ignore round-off in computing sizes of  $A^{(i)}$ 's and  $B^{(i)}$ 's.

### Step 1: There are large $\tilde{A}^{(i)}$ sets

Let  $t = h \lfloor N/(3h) \rfloor$ . We will eventually modify each of the sets  $\tilde{A}^{(i)}$  into sets  $A_1^{(i)}$  that are either of size  $t$  or  $t + h$ . Let  $N = (3q + r)h$  with  $r \in \{0, 1, 2\}$ . More precisely, the largest  $r$  sets  $\tilde{A}^{(i)}$  will be modified into sets  $A_1^{(i)}$  of size  $t + h$  and the smallest  $3 - r$  sets  $\tilde{A}^{(j)}$  will be modified into sets  $A_1^{(j)}$  of size  $t$ .

We will find, in  $\tilde{A}^{(1)} \cup \tilde{A}^{(2)} \cup \tilde{A}^{(3)}$ , (vertex-disjoint)  $h$ -stars. We need the following lemma, proven in Section 4.6.

**Lemma 4.4** *Let us be given  $\epsilon > 0$  and a positive integer  $M$ .*

- (1) *Let  $(A^{(1)}, A^{(2)}; E)$  be a bipartite graph such that every vertex in  $A^{(2)}$  is adjacent to at least  $d_1$  vertices in  $A^{(1)}$ . Furthermore,  $||A^{(i)}| - M| < \epsilon M$  and  $d_i < \epsilon M$  for  $i = 1, 2$ .*

*Provided  $\epsilon < ((h + 1)h)^{-1}$ , there is a family of vertex-disjoint copies of  $K_{1,h}$  such that  $\max\{0, d_1 - h + 1\}$  of them have centers in  $A^{(1)}$ .*

- (2) *Let  $(A^{(1)}, A^{(2)}, A^{(3)}; E)$  be a tripartite graph such that every vertex not in  $A^{(i)}$  is adjacent to at least  $d_i$  vertices in  $A^{(i)}$ , for  $i = 1, 2, 3$ . Furthermore,  $||A^{(i)}| - M| < \epsilon M$  and  $d_i < \epsilon M$  for  $i = 1, 2, 3$ .*

*Provided  $\epsilon < (2(h + 2)(h + 1)h)^{-1}$ , there is a family of vertex-disjoint copies of  $K_{1,h}$  such that  $\max\{0, d_i - h + 1\}$  of them have centers in  $A^{(i)}$  and leaves in  $A^{(i+1)}$  (index arithmetic is modulo 3).*

With our degree condition, we can guarantee that each vertex not in  $V^{(i)}$  is adjacent to at least  $|\tilde{A}^{(i)}| - t + h - 1$  vertices in  $\tilde{A}^{(i)}$ . So, we use Lemma 4.4(2) with  $d_i \geq |\tilde{A}^{(i)}| - t + h - 1$  to construct

the stars with the property that there are exactly enough centers in  $\tilde{A}^{(i)}$  such that, when removed, the resulting set has its size bounded above by either  $t$  or  $t + h$ , whichever is required. Place these centers into  $Z^{(i)}$ .

**Step 2: There are small  $\tilde{A}^{(i)}$  sets**

For a subgraph  $K_{1,h,h}$ , with  $h \geq 2$ , define the *center* to be the vertex that is adjacent to all others. We will also refer to the remaining vertices as *leaves*, although their degree is  $h + 1$ .

We will find, in  $B := \bigcup_{i=1}^3 (\tilde{B}^{(i)} \cup C^{(i)})$ , (vertex-disjoint) copies of  $K_{1,h,h}$  such that  $\max\{t + h - |\tilde{A}^{(i)}|, 0\}$  copies having its center vertex in  $B^{(i)}$  for the largest  $r$  sets  $\tilde{A}^{(i)}$  and such that  $t - |\tilde{A}^{(j)}|$  copies having the center vertex in  $B^{(j)}$  for the smallest  $3 - r$  sets  $\tilde{A}^{(j)}$ . This will be accomplished with Lemma 4.5, proven in Section 4.6.

**Lemma 4.5** *Given  $\delta > 0$ , there exists an  $\epsilon = \epsilon(\delta) > 0$  such that the following occurs:*

*Let  $(B^{(1)}, B^{(2)}, B^{(3)}; E)$  be a tripartite graph such that for all  $i \neq j$ , each vertex in  $B^{(i)}$  is adjacent to at least  $(1 - \epsilon)M$  vertices in  $B^{(j)}$ . Furthermore,  $||B^{(i)}| - 2M| < \epsilon M$ .*

*If  $(B^{(1)}, B^{(2)}, B^{(3)}; E)$  contains no copy of  $K_{1,h,h}$  with 1 vertex in  $B^{(1)}$ , and  $h$  vertices in each of  $B^{(2)}$  and  $B^{(3)}$ , then the graph  $(B^{(1)}, B^{(2)}, B^{(3)}; E)$  is  $\delta$ -approximately  $\Theta_{3 \times 2}(M)$ .*

Lemma 4.5 can be repeatedly applied at most  $\lceil \Delta_1(N/3) \rceil$  times, unless  $G$  is  $\Delta_2$ -approximately  $\Theta_{3 \times 3}(t)$ . Here, we will want  $\Delta_1 + 6\Delta_1^2 < \epsilon(\Delta_2)$ . Add the center vertices of the  $K_{1,h,h}$  subgraphs to the appropriate sets  $\tilde{A}^{(i)}$ .

Place vertices from  $C^{(i)}$  into the sets  $\tilde{A}^{(i)}$  so that  $A_1^{(i)}$  is of size  $t$  or  $t + h$ , for  $i = 1, 2, 3$  and that  $\sum_{i=1}^3 |A_1^{(i)}| = N$ . Relabel the modified sets  $\tilde{A}^{(i)}$  with  $A_1^{(i)}$ .

**Step 3: Finding a  $K_{h,h}$ -factor in  $B$**

Now we try to find a  $K_{h,h}$ -factor among the remaining vertices in  $B$  with the goal of matching them with the  $A_1^{(i)}$  vertices. There are, however, some adjustments that should be made.

- Vertices which are in copies of  $K_{1,h,h}$ , where the center vertex is in some  $A_1^{(i)}$ , will be in a specific copy of  $K_{h,h}$  in  $B$ .
- If  $v \in Z^{(i)}$  is the center of a  $K_{1,h}$  with leaves in  $A_k^{(i)}$ , then  $v$  will be assigned to  $B^{(j)}$ , where  $\{j\} = \{1, 2, 3\} \setminus \{i, k\}$ .

- Vertices  $v \in C^{(i)}$  will be assigned to  $B^{(j)}$  if  $v$  is adjacent to at least  $(2\Delta_1)(N/3)$  vertices in  $A^{(k)}$ . Since  $v \in C^{(i)}$  it will be assigned either to  $B^{(j)}$  or to  $B^{(k)}$ , where  $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$ .

This last statement results from the fact that if  $v \in C^{(i)}$ , then we may assume, without loss of generality, that  $v$  is adjacent to less than  $(1 - 2\Delta_1^2)(2N/3)$  vertices in, say,  $B^{(j)}$ . Hence,  $v$  is adjacent to at least  $(2\Delta_1^2)(N/3)$  vertices in  $A^{(j)}$  and at least  $(3\Delta_1/2)(N/3)$  vertices in  $A_1^{(i)}$ .

Moreover, we have that  $|C^{(i)}| \leq 9\Delta_1^2(N/3)$ ,  $|Z^{(i)}| \leq 6\Delta_1^2(N/3)$  and there are at most  $4\Delta_1^2(N/3)$  copies of  $K_{1,h,h}$  with the center vertex in a given  $A_1^{(i)}$ .

Lemma 4.6 is proven in Section 4.6.

**Lemma 4.6** *Let us be given  $\delta > 0$ . Then there exists an  $\epsilon = \epsilon(\delta) > 0$  and a positive integer  $t_0 = t_0(\delta)$  such that the following occurs:*

*Let there be positive integers  $t_1, t_2, t_3$  which are divisible by  $h$  and with  $|t_i - t_j| \in \{0, h\}$ , for all  $i, j \in \{1, 2, 3\}$  and  $t_1 > t_0$ . Let  $(B^{(1)}, B^{(2)}, B^{(3)}; E)$  be a tripartite graph such that for distinct indices  $i, j, k \in \{1, 2, 3\}$ ,  $|B^{(i)}| = t_j + t_k$ . For all  $i \neq j$ , each vertex in  $B^{(i)}$  is adjacent to at least  $(1 - \epsilon)t_1$  vertices in  $B^{(j)}$ . We attempt to find a  $K_{h,h}$ -factor in the graph induced by  $(B^{(1)}, B^{(2)}, B^{(3)}; E)$  with certain restrictions:*

*For each pair  $(B^{(i)}, B^{(j)})$ , there are at most  $\epsilon t_1$  copies of  $K_{h,h}$  which must be part of any factor. For each  $B^{(i)}$ , there are at most  $\epsilon t_1$  vertices with the following property:  $v$  can only be in copies of  $K_{h,h}$  in the pair  $(B^{(i)}, B^{(j)})$  and  $v$  is adjacent to at least  $(1 - \epsilon)t_1$  vertices in  $B^{(i)}$ .*

*If such a factor cannot be found, then, without loss of generality, the graph induced by  $(B^{(1)}, B^{(2)}, B^{(3)}; E)$  can be partitioned such that  $B^{(i)} = B^{(i)}[1] + B^{(i)}[2]$ ,  $|B^{(i)}[1]| = t_1$  for  $i = 1, 2, 3$  and  $d(B^{(j)}[1], B^{(2)}[1]) \leq \delta$  and  $d(B^{(j)}[2], B^{(2)}[2]) \leq \delta$  for  $j = 1, 3$ .*

Then, match vertices in  $C^{(i)}$  that are assigned to  $B^{(j)}$  with  $h$  typical neighbors in  $B^{(j)}[i]$  and those with  $h - 1$  typical neighbors in  $B^{(i)}[j]$ . Finally, place the vertices that were moved into copies of  $K_{h,h,h}$ . All of these will be removed, allowing us to apply Lemma 4.6. If the appropriate  $K_{h,h}$ -factor cannot be found, then we are in the case of Part 2. The diagram that defines that case is in Figure 2.

#### Step 4: Completing the $K_{h,h,h}$ -factor

We use Proposition 4.7, which allows us to complete a  $K_{h,h}$ -factor into a  $K_{h,h,h}$ -factor. The proof follows easily from König-Hall and is in Section 4.6.

**Proposition 4.7** *Let  $h \geq 1$ .*

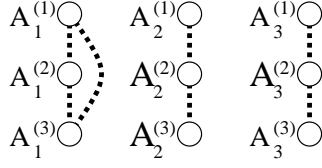


Figure 2: The graph that defines Part 2. A dotted line represents a sparse pair.

- (1) Let  $G = (V^{(1)}, V^{(2)}; E)$  be a bipartite graph with  $|V^{(1)}| = |V^{(2)}| = M$ ,  $h$  divides  $M$ , and each vertex is adjacent to at least  $(1 - \frac{1}{2h^2})M$  vertices in the other part. Then, we can find a  $K_{h,h}$ -factor in  $G$ .
- (2) Let  $G = (V^{(1)}, V^{(2)}, V^{(3)}; E)$  be a tripartite graph with  $|V^{(1)}| = |V^{(2)}| = |V^{(3)}| = M$ ,  $h$  divides  $M$ , and each vertex is adjacent to at least  $(1 - \frac{1}{4h^2})M$  vertices in each of the other parts. Furthermore, let there be a  $K_{h,h}$ -factor in  $(V^{(2)}, V^{(3)})$ . Then, we can extend it into a  $K_{h,h,h}$ -factor in  $G$ .

This allows us to find  $K_{h,h,h}$ -factors in each of  $(A_1^{(1)}, B^{(2)}[3], B^{(3)}[2])$ ,  $(A_1^{(2)}, B^{(1)}[3], B^{(3)}[1])$  and  $(A_1^{(3)}, B^{(1)}[2], B^{(2)}[1])$  which completes the  $K_{h,h,h}$ -factor in  $G$ .

## 4.2 Part 2: $G$ is approximately the graph in Figure 2

**Remark.** In this part, we must deal with the fact that the sets  $A_2^{(2)}$  and  $A_3^{(2)}$  may have close to the same number of vertices, but that number is not divisible by  $h$ . Much more work needs to be done in order to modify these sets so that their sizes become divisible by  $h$ . We think it is easier to see the basic arguments in the relatively shorter Part 1 before addressing the specific issues raised in Part 2.

Recall that each vertex is adjacent to at least  $h \lceil \frac{2N}{3h} \rceil + h - 1$  vertices in each of the other pieces of the partition. Again, let  $t = h \lfloor N/(3h) \rfloor$ . We will transform the graph that is  $\Delta_2$ -approximately a graph defined by Figure 2 with the vertices corresponding to sets of size  $N/3$ .

Before we begin, we must examine the behavior of  $(A_2^{(1)} \cup A_3^{(1)}, A_2^{(3)} \cup A_3^{(3)})$ . If this is  $\Delta_5$ -approximately  $\Theta_{2 \times 2}(N/3)$ , then call the dense pairs  $(E^{(1)}, E^{(3)})$  and  $(F^{(1)}, F^{(3)})$ . Otherwise, coincidence can only occur in either  $V^{(1)}$  or  $V^{(3)}$ , but not both. Without loss of generality, we will assume that if there is such a coincidence, then it occurs in  $V^{(1)}$ .

We say that these pairs **coincide** with the sets  $A_j^{(i)}$  if the typical vertices of, say  $A_2^{(3)}$ , have small intersection with those of  $F^{(3)}$ . We will determine the quantity that constitutes “small” later. If  $(E^{(1)}, E^{(3)})$  and  $(F^{(1)}, F^{(3)})$  both coincide with  $(A_2^{(1)}, A_3^{(3)})$  and  $(A_3^{(1)}, A_2^{(3)})$ , then  $G$  is a graph

that is approximately  $\Theta_{3 \times 3}(N/3)$  (Section 4.3). If  $(E^{(1)}, E^{(3)})$  and  $(F^{(1)}, F^{(3)})$  both coincide with  $(A_2^{(1)}, A_2^{(3)})$  and  $(A_3^{(1)}, A_3^{(3)})$ , then approximately  $\Gamma_3(N/3)$  (Section 4.4). Otherwise, coincidence can only occur in either  $V^{(1)}$  or  $V^{(3)}$ , but not both. Without loss of generality, we will assume that if there is such a coincidence, then it occurs in  $V^{(1)}$ .

Let  $V^{(i)} = A_1^{(i)} + A_2^{(i)} + A_3^{(i)} + C^{(i)}$ , such that each  $A_j^{(i)}$  has size  $(1 - 3\Delta_2^{2/3})t$  and  $(1 + 3\Delta_2^{2/3})t$  and each vertex in  $A_j^{(i)}$  is adjacent to at least  $\theta t$  vertices in each set  $A_{j'}^{(i')}$  for which one of the following occurs:

- $i = 2$  and  $j' \neq j$
- $i \in \{1, 3\}$ ,  $j = 1$  and  $j' \neq j$
- $i \in \{1, 3\}$ ,  $i' = 2$  and  $j \in \{2, 3\}$
- $i \in \{1, 3\}$ ,  $i' = 4 - i$ ,  $j \in \{2, 3\}$  and  $j' = 1$

In other words, the vertices in  $A_j^{(i)}$  are the ones that are typical according to the rules established by Figure 2. In addition, if, say  $A_2^{(1)}$  coincides with  $E^{(1)}$ , then every vertex in  $A_2^{(1)}$  is adjacent to at least  $\theta t$  vertices in  $E^{(3)}$  and vice versa. If there is no coincidence, then let  $E^{(1)}$  and  $E^{(3)}$  be redefined so that every vertex in  $E_1$  is adjacent to at least  $\theta t$  vertices in  $E^{(3)}$  and vice versa. Similarly for  $(F^{(1)}, F^{(3)})$ .

Each vertex  $c \in C^{(2)}$  has the property that, for all  $j \in \{1, 2, 3\}$  and distinct  $i', i'' \in \{1, 3\}$ , if  $c$  is adjacent to fewer than  $\Delta_3 t$  vertices in  $A_j^{(i')}$ , then  $c$  is adjacent to at least  $\Delta_3 t$  vertices in  $A_j^{(i'')}$ .

Let  $i \in \{1, 3\}$ , each vertex  $c \in C^{(i)}$  has the property that, for all  $j \in \{1, 2, 3\}$ ,  $c$  cannot be adjacent to fewer than  $\Delta_3 t$  vertices in either  $A_2^{(2)}$  or  $A_3^{(2)}$ . Also,  $c$  cannot be adjacent to fewer than  $\Delta_3 t$  vertices in both  $A_1^{(2)}$  and  $A_1^{(4-i)}$  or both  $A_2^{(2)}$  and  $F^{(4-i)}$  (if it exists) or both  $A_3^{(2)}$  and  $E^{(4-i)}$  (if it exists).

Trivially, each vertex in  $V^{(i)}$  is adjacent to at least  $(1/2 - \Delta_3)t$  vertices in at least two of  $\{A_1^{(i')}, A_2^{(i')}, A_3^{(i')}\}$  and in at least two of  $\{A_1^{(i'')}, A_2^{(i'')}, A_3^{(i'')}\}$ , where  $i', i''$  are distinct members of  $\{1, 2, 3\} \setminus \{i\}$ . This is particularly important for vertices in  $C^{(i)}$ .

### Step 1: Ensuring small $A_j^{(i)}$ sets

First, take each triple  $(A_j^{(1)}, A_j^{(2)}, A_j^{(3)})$ ,  $j = 1, 2, 3$ , and construct disjoint copies of stars so that there are at most  $t$  non-center vertices in each set  $A_j^{(i)}$ . As in Part 1, we use the fact that every vertex is adjacent to at least  $h \lceil \frac{2N}{3h} \rceil + h - 1$  vertices in each of the other parts as well as Lemma 4.4. For  $i, j = 1, 2, 3$ , place  $|A_j^{(i)}| - t$  centers from  $A_j^{(i)}$  into a set  $Z^{(i)}$ .

### Step 2: Fixing the size of $A_j^{(i)}$ sets

We have sets  $A_j^{(i)}$  which have  $|A_j^{(i)}| \leq t$  and the remaining vertices are in sets  $C^{(i)} \cup Z^{(i)}$ . Since  $N$  is divisible by  $h$ , we can place the vertices  $C^{(i)} \cup Z^{(i)}$  arbitrarily into sets  $A_1^{(i)}$ ,  $A_2^{(i)}$  and  $A_3^{(i)}$  so that the resulting sets  $A_j^{(i)}$  have cardinality  $t$  or  $t + h$  and for  $j = 1, 2, 3$ ,

$$|A_j^{(1)}| + |A_j^{(2)}| + |A_j^{(3)}| = N.$$

For this purpose, if  $N/h \cong 1 \pmod{3}$ , add  $h$  vertices to each of  $A_2^{(1)}$ ,  $A_1^{(2)}$  and  $A_3^{(3)}$ . If  $N/h \cong 2 \pmod{3}$ , add  $h$  vertices to all sets  $A_j^{(i)}$ , except  $A_2^{(1)}$ ,  $A_1^{(2)}$  and  $A_3^{(3)}$ .

### Step 3: Partitioning the sets

We will partition each set  $A_j^{(i)}$  into two pieces, as close as possible to equal size, but which have size divisible by  $h$ . This must have the property that a typical vertex in  $A_j^{(i)}$  has at least  $\left(1 - 2\Delta_4 - 6\Delta_2^{2/3}\right)(t/2)$  neighbors in each piece of the partition of  $A_{j'}^{(i')}$ ,  $i' \neq i$ ,  $j' \neq j$ . Moreover, if a vertex has degree at least  $\Delta_3 t$  in a set, it has degree at least  $(\Delta_3/3)(t/2)$  in each of the two partitions. Such a partition exists, almost surely, provided  $N$  is large enough, if the partition is random.

Assign to each part a permutation,  $\sigma \in \Sigma_3$ , which assigns  $j = \sigma(i)$ . ( $\Sigma_3$  denotes the symmetric group that permutes the elements of  $\{1, 2, 3\}$ .) Each part assigned to  $\sigma$  will be the same size.

### Step 4: Assigning vertices

The former  $C^{(i)}$  vertices, as well as star-leaves and star-centers, may only be able to form a  $K_{h,h,h}$  with respect to one particular permutation.

For example, consider a vertex  $c$  which had been in  $C^{(1)}$  but is now in  $A_1^{(1)}$ . Then, for either the pair  $(A_2^{(2)}, A_3^{(3)})$  or the pair  $(A_3^{(2)}, A_2^{(3)})$ , the vertex  $c$  is adjacent to at least  $(1/2 - \Delta_3)t$  in one set and at least  $\Delta_3 t$  vertices in the other; otherwise, it would have been a typical vertex in  $A_1^{(1)}$ ,  $A_2^{(1)}$  or  $A_3^{(1)}$ .

Assume that  $c$  is adjacent to at least  $\Delta_3 t$  vertices in  $A_3^{(2)}$  and at least  $(1/2 - \Delta_3)t$  vertices in  $A_2^{(3)}$ . In this case, if  $c$  were placed into the partition corresponding to the identity permutation, then exchange  $c$  with a typical vertex in the partition assigned to  $(23)$ , using cycle notation of permutations.

In a similar fashion, if there is a star with center in, say  $A_2^{(1)}$ , and leaves in, say  $A_1^{(2)}$ , then we will use it to form a  $K_{h,h,h}$  with respect to the permutation  $(12) \in \Sigma_3$ . Again, if any such leaf or center was in the wrong partition, exchange it with a typical vertex in the other partition.

The number of leaves in any set is at most  $2h \left(6\Delta_2^{2/3}t + h\right)$  and the number of centers is at most  $2 \left(6\Delta_2^{2/3}t + h\right)$ , the number of  $C^{(i)}$  vertices is at most  $9\Delta_2^{2/3}t$ . So, if  $N$  is large enough, the total number of typical vertices in any  $A_j^{(i)}$  which were exchanged is at most  $2(12h + 21)\Delta_2^{2/3}t + 4h^2 + 4h$ .

With the partition established and the  $C^{(i)}$ ,  $Z^{(i)}$  and leaf vertices in the proper part, we consider the triple formed by three sets:

- $A_1^{(2)}$ , which will also be denoted  $\tilde{S}^{(2)}$
- the union of the piece of  $A_2^{(1)}$  corresponding to (12) and the piece of  $A_3^{(1)}$  corresponding to (132), denoted  $\tilde{S}^{(1)}$ , and
- the union of the piece of  $A_2^{(3)}$  corresponding to (132) and the piece of  $A_3^{(3)}$  corresponding to (12), denoted  $\tilde{S}^{(3)}$ .

Let the graph induced by the triple  $(\tilde{S}^{(1)}, \tilde{S}^{(2)}, \tilde{S}^{(3)})$  be denoted  $\tilde{\mathcal{S}}$ .

### Step 5: Finding a $K_{h,h,h}$ cover in $\tilde{\mathcal{S}}$

Let  $t_0 = |A_1^{(2)}|$ . First, take each  $K_{1,h}$  in  $\mathcal{S}'$  and complete it to form disjoint copies of  $K_{h,h,h}$ , using unexchanged typical vertices. This can be done if  $\Delta_4$  is small enough. Remove all such  $K_{h,h,h}$ 's containing stars.

Second, take each  $c$  which had been a member of some  $C^{(i)}$  and use it to complete a  $K_{h,h,h}$ . We can guarantee, because of the random partitioning, that  $c$  is adjacent to at least  $(\Delta_3/3)t_0$  vertices in one set and  $(1/3 - 2\Delta_3)t_0$  vertices in the other. Without loss of generality, let  $c \in \tilde{S}^{(1)}$  with degree at least  $(\Delta_3/3)t_0$  in  $\tilde{S}^{(2)}$  and at least  $(1/4 - 2\Delta_3)t_0$  in  $\tilde{S}^{(3)}$ . Since  $\Delta_3 \gg \Delta_2$ , we can guarantee  $h$  neighbors of  $c$  in  $\tilde{S}^{(2)}$  among unexchanged typical vertices and, if  $\Delta_3 \ll \Delta_4 \ll 1$ , then  $h$  common neighbors of those among unexchanged typical vertices in  $N(c) \cap \tilde{S}^{(3)}$ . Finally,  $\Delta_4 \ll h^{-1}$  implies this  $K_{h,h}$  has at least  $h - 1$  more common neighbors in  $\tilde{S}^{(1)}$ . This is our  $K_{h,h,h}$  and we can remove it. Do this for all former members of a  $C^{(i)}$ .

Third, take each exchanged typical vertex and put it into a  $K_{h,h,h}$  and remove it. Throughout this process, we have removed at most  $C_h \sqrt{\Delta_2} \times t_0$  vertices where  $C_h$  is a constant depending only on  $h$ . What remains are three sets of the same size,  $t' \geq (1 - C_h \sqrt{\Delta_2})t_0$ , with each vertex in  $\tilde{S}^{(1)}$  adjacent to at least, say  $(1/2 - 2\Delta_4)t'$ , vertices in  $\tilde{S}^{(3)}$  and vice versa. Each vertex in  $\tilde{S}^{(1)}$  and in  $\tilde{S}^{(3)}$  is adjacent to at least  $(1/2 - 2\Delta_4)t'$  vertices in  $\tilde{S}^{(2)}$  and each vertex in  $\tilde{S}^{(2)}$  is adjacent to at least  $(1/2 - 2\Delta_4)t'$  vertices in  $\tilde{S}^{(1)}$  and in  $\tilde{S}^{(3)}$ .

Lemma 4.8, from [26], shows that we can find a factor of  $(\tilde{S}^{(1)}, \tilde{S}^{(3)})$  with vertex-disjoint copies of  $K_{h,h}$  unless  $(\tilde{S}^{(1)}, \tilde{S}^{(3)})$  is approximately  $\Theta_{2 \times 2}(N/6)$ . In that case, find the factor and finish to

form a factor of  $\tilde{\mathcal{S}}$  of vertex-disjoint copies of  $K_{h,h,h}$  via König-Hall.

**Lemma 4.8 (Z. [26])** *For every  $\epsilon > 0$  and integer  $h \geq 1$ , there exists an  $\alpha > 0$  and an  $N_0$  such that the following holds. Suppose that  $N > N_0$  is divisible by  $h$ . Then every bipartite graph  $G = (A, B; E)$  with  $|A| = |B| = N$  and  $\delta(G) \geq (1/2 - \alpha)N$  either contains a  $K_{h,h}$ -factor, or contains  $A' \subseteq A$ ,  $B' \subseteq B$  such that  $|A'| = |B'| = N/2$  and  $d(A, B) \leq \epsilon$ .*

Lemma 4.9 states, in particular, that if a random partition results in  $(\tilde{\mathcal{S}}^{(1)}, \tilde{\mathcal{S}}^{(3)})$  being approximately  $\Theta_{2 \times 2}(N/6)$  with high probability, then  $(A_2^{(1)} \cup A_3^{(1)}, A_2^{(3)} \cup A_3^{(3)})$  is approximately  $\Theta_{2 \times 2}(N/3)$ . The proof of Lemma 4.9 follows from similar arguments to those in the proof of Lemma 3.3 of [19] and in Section 3.3.1 of [20] so we omit it.

**Lemma 4.9** *For every  $\epsilon > 0$  and integer  $h \geq 1$ , there exists a  $\beta > 0$  and positive integer  $t_0$  such that if  $t \geq t_0$  the following holds. Let  $(A, B)$  be a bipartite graph such that  $|A|, |B| \in \{2t - h, 2t, 2t + h\}$  with minimum degree at least  $(1 - \epsilon)t$  and is minimal with respect to this condition. Let  $A' \subset A$ ,  $B' \subset B$ ,  $|A'| = |B'| = t$  be chosen uniformly at random. If*

$$\Pr\{(A', B') \text{ contains a subpair with density at most } \epsilon\} \geq 1/4$$

*then  $(A, B)$  is  $\beta$ -approximately  $\Theta_{2 \times 2}(t)$ .*

We can, therefore, assume the existence of  $(E^{(1)}, E^{(3)})$  and  $(F^{(1)}, F^{(3)})$ . Otherwise, Lemmas 4.8 and 4.9 imply that  $\tilde{\mathcal{S}}$  has a  $K_{h,h,h}$ -factor.

As a result, recall that we let the typical vertices in the dense pairs in  $(A_2^{(1)} \cup A_3^{(1)}, A_2^{(3)} \cup A_3^{(3)})$  be denoted  $(E^{(1)}, E^{(3)})$  and  $(F^{(1)}, F^{(3)})$ . If the dense pairs do not coincide, then we will work to ensure that  $|E^{(1)} \cap \tilde{\mathcal{S}}^{(1)}| = |E^{(3)} \cap \tilde{\mathcal{S}}^{(3)}|$  and  $|F^{(1)} \cap \tilde{\mathcal{S}}^{(1)}| = |F^{(3)} \cap \tilde{\mathcal{S}}^{(3)}|$  and both are divisible by  $h$ . Do this by moving vertices from  $(A_2^{(1)} \cap E^{(1)}) \setminus \tilde{\mathcal{S}}^{(1)}$  into  $(A_2^{(1)} \cap E^{(1)}) \cap \tilde{\mathcal{S}}^{(1)}$  and move the same number from  $(A_2^{(1)} \cap F^{(1)}) \cap \tilde{\mathcal{S}}^{(1)}$  into  $(A_2^{(1)} \cap F^{(1)}) \setminus \tilde{\mathcal{S}}^{(1)}$ . In addition, move vertices from  $(A_2^{(3)} \cap E^{(3)}) \setminus \tilde{\mathcal{S}}^{(3)}$  into  $(A_2^{(3)} \cap E^{(3)}) \cap \tilde{\mathcal{S}}^{(3)}$  and move the same number from  $(A_2^{(3)} \cap F^{(3)}) \cap \tilde{\mathcal{S}}^{(3)}$  into  $(A_2^{(3)} \cap F^{(3)}) \setminus \tilde{\mathcal{S}}^{(3)}$ .

This can be done unless one of the intersections  $A_j^{(i)} \cap E^{(i)}$  or  $A_j^{(i)} \cap F^{(i)}$  is too small. This implies the coincidence that we discussed at the beginning of this part. But then, we have guaranteed that the remaining vertices of  $A_2^{(1)}$  are not only typical in that set but also typical in  $E^{(1)}$ . The same is true of  $A_3^{(1)}$  and  $F^{(1)}$ .

Now, we want to move vertices in  $V^{(3)}$  to ensure that  $|E^{(3)} \cap \tilde{\mathcal{S}}^{(3)}| = |A_2^{(1)} \cap \tilde{\mathcal{S}}^{(1)}|$  and  $|F^{(3)} \cap \tilde{\mathcal{S}}^{(3)}| = |A_3^{(1)} \cap \tilde{\mathcal{S}}^{(1)}|$ . Note that we have ensured that both  $|A_2^{(1)} \cap \tilde{\mathcal{S}}^{(1)}|$  and  $|A_3^{(1)} \cap \tilde{\mathcal{S}}^{(1)}|$  are divisible by  $h$  and approximately  $N/6$ .

We can do this as follows: Move vertices from  $E^{(3)} \cap A_2^{(3)} \setminus \tilde{\mathcal{S}}^{(3)}$  to  $(E^{(3)} \cap A_2^{(3)}) \cap \tilde{\mathcal{S}}^{(3)}$  and move the same amount from  $(F^{(3)} \cap A_2^{(3)}) \cap \tilde{\mathcal{S}}^{(3)}$  to  $(F^{(3)} \cap A_2^{(3)}) \setminus \tilde{\mathcal{S}}^{(3)}$ . Also move vertices from  $(E^{(3)} \cap A_3^{(3)}) \setminus \tilde{\mathcal{S}}^{(3)}$  to  $(E^{(3)} \cap A_3^{(3)}) \cap \tilde{\mathcal{S}}^{(3)}$  and move the same amount from  $(F^{(3)} \cap A_3^{(3)}) \cap \tilde{\mathcal{S}}^{(3)}$  to  $(F^{(3)} \cap A_3^{(3)}) \setminus \tilde{\mathcal{S}}^{(3)}$ . Since none of the intersections are small, this is possible. Complete this to vertex-disjoint copies of  $K_{h,h,h}$  in  $\tilde{\mathcal{S}}$  by Proposition 4.7.

### Step 6: Completing the $K_{h,h,h}$ -factor in $G$

Now that we have found a  $K_{h,h,h}$  that corresponds permutations (12) and (132), we consider permutations in  $\Sigma_3$ . For a  $\sigma \in \Sigma_3 \setminus \{(12), (132)\}$ , let  $\mathcal{S}(\sigma) \stackrel{\text{def}}{=} (S_{\sigma(1)}^{(1)}, S_{\sigma(2)}^{(2)}, S_{\sigma(3)}^{(3)})$  be a triple of parts formed by the random partitioning after the exchange of vertices has taken place. The set  $S_{\sigma(i)}^{(i)}$  is a subset of  $A_{\sigma(i)}^{(i)}$ . We have also ensured that  $s_\sigma \stackrel{\text{def}}{=} |S_{1,\sigma(1)}| = |S_{2,\sigma(2)}| = |S_{3,\sigma(3)}|$  and  $s_\sigma$  is divisible by  $h$ . It is now easy to ensure that this triple contains a  $K_{h,h,h}$ -factor:

First, take each star in  $\mathcal{S}(\sigma)$  and complete it to form disjoint copies of  $K_{h,h,h}$ , using unexchanged typical vertices. This can be done if  $\Delta_4$  is small enough. Remove all such  $K_{h,h,h}$ 's containing stars.

Second, take each  $c$  which had been a member of some  $C^{(i)}$  and use it to complete a  $K_{h,h,h}$ . We can guarantee, because of the random partitioning, that  $c$  is adjacent to at least  $(\Delta_3/3)s_\sigma$  vertices in one set and  $(2/3 - 2\Delta_3)s_\sigma$  vertices in the other. Without loss of generality, let  $c \in S_{\sigma(1)}^{(1)}$  with degree at least  $(\Delta_3/3)s_\sigma$  in  $S_{\sigma(2)}^{(2)}$  and at least  $(1/2 - 2\Delta_3)s_\sigma$  in  $S_{\sigma(3)}^{(3)}$ . Since  $\Delta_3 \gg \Delta_2$ , we can guarantee  $h$  neighbors of  $c$  in  $S_{\sigma(2)}^{(2)}$  among unexchanged typical vertices and, if  $\Delta_3 \ll \Delta_4 \ll 1$ , then  $h$  common neighbors of those among unexchanged typical vertices in  $N(c) \cap S_{\sigma(3)}^{(3)}$ . Finally,  $\Delta_4 \ll h^{-1}$  implies this  $K_{h,h}$  has at least  $h - 1$  more common neighbors in  $S_{\sigma(1)}^{(1)}$ . This is our  $K_{h,h,h}$  and we can remove it. Do this for all former members of a  $C^{(i)}$ .

Finally, take each exchanged typical vertex and put it into a  $K_{h,h,h}$  and remove it. Throughout this process, we have removed at most  $C_h \sqrt{\Delta_2} \times s_\sigma$  vertices where  $C_h$  is a constant depending only on  $h$ . What remains are three sets of the same size,  $s' \geq (1 - C_h \sqrt{\Delta_2})s_\sigma$ , with each vertex adjacent to at least, say  $(1 - 2\Delta_4)s'$ , vertices in each of the other parts. If  $N$  is large enough, then we can use the Blow-up Lemma or Proposition 4.7(2) to complete the factor of  $\mathcal{S}(\sigma)$  by copies of  $K_{h,h,h}$ .

### 4.3 Part 3a: $G$ is approximately $\Theta_{3 \times 3}(\lfloor N/3 \rfloor)$

Figure 3 defines the case  $\Theta_{3 \times 3}(\lfloor N/3 \rfloor)$  where sets that are connected with a dotted line are sparse.

We will assume for this part that each vertex is adjacent to at least  $h \lfloor \frac{2N}{3h} \rfloor + h - 1$  vertices in each of the other pieces of the partition. Again, let  $t = h \lfloor N/(3h) \rfloor$ .

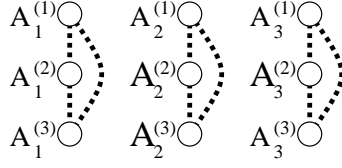


Figure 3: The graph that defines Part 2. A dotted line represents a sparse pair.

We will transform the  $\Delta_2$ -approximately  $\Theta_{3 \times 3}(\lfloor N/3 \rfloor)$  by partitioning  $V^{(i)}$ ,  $i = 1, 2, 3$ , into four sets, as follows:  $V^{(i)} = A_1^{(i)} + A_2^{(i)} + A_3^{(i)} + C^{(i)}$ , such that each  $A_j^{(i)}$  has size between  $(1 - \sqrt{\Delta_2})t$  and  $(1 + \sqrt{\Delta_2})t$  and each vertex in  $A_j^{(i)}$  is adjacent to at least  $\theta t$  vertices in each set  $A_{j'}^{(i')}$  where  $i' \neq i$  and  $j' \neq j$ .

Each vertex  $c \in C^{(i)}$  has the property that, for all  $j \in \{1, 2, 3\}$  and distinct  $i', i'' \in \{1, 2, 3\} \setminus \{i\}$ , if  $c$  is adjacent to fewer than  $\Delta_3 t$  vertices in  $A_j^{(i')}$ , then  $c$  is adjacent to at least  $\Delta_3 t$  vertices in  $A_j^{(i'')}$ ; otherwise  $c$  is in some set  $A_j^{(i)}$ . Furthermore,  $c$  is adjacent to at least  $(1/2 - \Delta_3)t$  vertices in at least two of  $\{A_1^{(i')}, A_2^{(i')}, A_3^{(i')}\}$  and in at least two of  $\{A_1^{(i'')}, A_2^{(i'')}, A_3^{(i'')}\}$ .

### Step 1: Ensuring small $A_j^{(i)}$ sets

First, take each triple  $(A_j^{(1)}, A_j^{(2)}, A_j^{(3)})$ ,  $j = 1, 2, 3$ , and construct disjoint copies of stars so that there are at most  $t$  non-center vertices in each set  $A_j^{(i)}$ . We use the fact that every vertex is adjacent to at least  $h \lceil \frac{2N}{3h} \rceil + h - 1$  vertices in each of the other parts as well as Lemma 4.4. For  $i, j = 1, 2, 3$ , place  $|A_j^{(i)}| - t$  centers from  $A_j^{(i)}$  into a set  $Z^{(i)}$ .

### Step 2: Fixing the size of $A_j^{(i)}$ sets

We have sets  $A_j^{(i)}$  which have  $|A_j^{(i)}| \leq t$  and the remaining vertices are in sets  $C^{(i)} \cup Z^{(i)}$ . Since  $N$  is divisible by  $h$ , we can place the vertices  $C^{(i)} \cup Z^{(i)}$  arbitrarily into sets  $A_1^{(i)}$ ,  $A_2^{(i)}$  and  $A_3^{(i)}$  so that the resulting sets  $A_j^{(i)}$  have cardinality  $t$  or  $t + h$  and for  $j = 1, 2, 3$ ,

$$|A_j^{(1)}| + |A_j^{(2)}| + |A_j^{(3)}| = N.$$

For this purpose, we could place these vertices first to ensure that all  $|A_j^{(i)}|$  become of size exactly  $t$ . If  $N = 3th + h$  then, for  $i = 1, 2, 3$ , add all of the remaining  $C^{(i)} \cup Z^{(i)}$  to  $A_i^{(i)}$ . If  $N = 3th + 2h$  then, for  $i = 1, 2, 3$ , add all of the remaining  $C^{(i)} \cup Z^{(i)}$  to  $A_j^{(i)}$ ,  $j \neq i$ .

### Step 3: Partitioning the sets

We will randomly partition each set  $A_j^{(i)}$  into two pieces, as close as possible to equal size but which have size divisible by  $h$ , and assign them to a permutation,  $\sigma \in \Sigma_3$ , which assigns  $j = \sigma(i)$ . ( $\Sigma_3$  denotes the symmetric group that permutes the elements of  $\{1, 2, 3\}$ .) Each part assigned to  $\sigma$  will be the same size. We call a vertex in  $A_j^{(i)}$  a **typical vertex** if it was not in  $C^{(i)}$  and is neither a star-leaf nor a star-center.

Note that a typical vertex in  $A_j^{(i)}$  has at least  $(1 - 2\Delta_4 - 2\sqrt{\Delta_2})(t/2)$  neighbors in each piece of the partition of  $A_{j'}^{(i')}$ ,  $i' \neq i$ ,  $j' \neq j$ , almost surely – provided  $N$  is large enough and the partition was as equitable as possible. Moreover, if a vertex has degree at least  $\Delta_3 t$  in a set, it has degree at least  $(\Delta_3/3)(t/2)$  in each of the two partitions.

#### Step 4: Assigning vertices

The former  $C^{(i)}$  vertices, as well as star-leaves and star-centers may only be able to form a  $K_{h,h,h}$  with respect to one particular permutation.

For example, consider a vertex  $c$  which had been in  $C^{(1)}$  but is now in  $A_1^{(1)}$ . Then, for either the pair  $(A_2^{(2)}, A_3^{(3)})$  or the pair  $(A_3^{(2)}, A_2^{(3)})$ , the vertex  $c$  is adjacent to at least  $(1/2 - \Delta_3)t$  in one set and at least  $\Delta_3 t$  vertices in the other. It is easy to see that, since  $\Delta_2 \ll \Delta_3$ , that if this were not true, then it would have been possible to place  $c$  into one of the sets  $A_1^{(1)}$ ,  $A_2^{(1)}$  or  $A_3^{(1)}$ .

Assume that  $c$  is adjacent to at least  $\Delta_3 t$  vertices in  $A_3^{(2)}$  and at least  $(1/2 - \Delta_3)t$  vertices in  $A_2^{(3)}$ . In this case, if  $c$  were placed into the partition corresponding to the identity permutation, then exchange  $c$  with a typical vertex in the partition assigned to  $(23)$ , using cycle notation of permutations.

In a similar fashion, if there is a star with center in, say  $A_2^{(1)}$ , and leaves in, say  $A_1^{(2)}$ , then we will form a  $K_{h,h,h}$  with respect to the permutation  $(12) \in \Sigma_3$ . Again, if any such leaf or center was in the wrong partition, exchange it with a typical vertex in the other partition.

The number of leaves in any set is at most  $2h(\sqrt{\Delta_2}t + h)$  and the number of centers is at most  $2(\sqrt{\Delta_2}t + h)$ , the number of  $C^{(i)}$  vertices is at most  $3\sqrt{\Delta_2}t$ . So, if  $N$  is large enough, the total number of typical vertices in any  $A_j^{(i)}$  which were exchanged is at most  $(2h + 6)\sqrt{\Delta_2}t$ .

#### Step 5: Completing the cover

For some  $\sigma \in \Sigma_3$ , let  $\mathcal{S}(\sigma) \stackrel{\text{def}}{=} (S_{\sigma(1)}^{(1)}, S_{\sigma(2)}^{(2)}, S_{\sigma(3)}^{(3)})$  be a triple of parts formed by the random partitioning after the exchange has taken place. The set  $S_{\sigma(i)}^{(i)}$  is a subset of  $A_{\sigma(i)}^{(i)}$ . We have also ensured in Step 3 that  $s_\sigma \stackrel{\text{def}}{=} |S_{\sigma(1)}^{(1)}| = |S_{\sigma(2)}^{(2)}| = |S_{\sigma(3)}^{(3)}|$  and  $s_\sigma$  is divisible by  $h$ . It is now easy to ensure that this triple contains a  $K_{h,h,h}$ -factor:

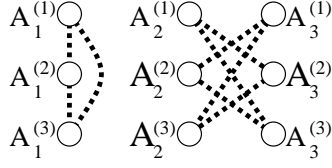


Figure 4: The graph that defines Part 3b. A dotted line represents a sparse pair.

First, take each star in  $\mathcal{S}(\sigma)$  and complete it to form disjoint copies of  $K_{h,h,h}$ , using unexchanged typical vertices. This can be done if  $\Delta_4$  is small enough. Remove all such  $K_{h,h,h}$ 's containing stars.

Second, take each  $c$  which had been a member of some  $C^{(i)}$  and use it to complete a  $K_{h,h,h}$ . We can guarantee, because of the random partitioning, that  $c$  is adjacent to at least  $(\Delta_3/3)s_\sigma$  vertices in one set and  $(2/3 - 2\Delta_3)s_\sigma$  vertices in the other. Without loss of generality, let  $c \in S_{\sigma(1)}^{(1)}$  with degree at least  $(\Delta_3/3)s_\sigma$  in  $S_{\sigma(2)}^{(2)}$  and at least  $(1/2 - 2\Delta_3)s_\sigma$  in  $S_{\sigma(3)}^{(3)}$ . Since  $\Delta_3 \gg \Delta_2$ , we can guarantee  $h$  neighbors of  $c$  in  $S_{\sigma(2)}^{(2)}$  among unexchanged typical vertices and, since  $\Delta_3 \ll \Delta_4 \ll 1$ ,  $h$  common neighbors of those among unexchanged typical vertices in  $N(c) \cap S_{\sigma(3)}^{(3)}$ . Finally,  $\Delta_4 \ll h^{-1}$  implies this  $K_{h,h}$  has at least  $h - 1$  more common neighbors in  $S_{\sigma(1)}^{(1)}$ . This is our  $K_{h,h,h}$  and we can remove it. Do this for all former members of a  $C^{(i)}$ .

Finally, take each exchanged typical vertex and put it into a  $K_{h,h,h}$  and remove it. Throughout this process, we have removed at most  $\Delta_2^{1/3}s_\sigma$  vertices if  $\Delta_2$  is small enough. What remains are three sets of the same size,  $s' \geq (1 - \Delta_2^{1/3})s_\sigma$ , with each vertex adjacent to at least, say  $(1 - 2\Delta_4)s'$ , vertices in each of the other parts. If  $N$  is large enough, then we can use the Blow-up Lemma or Proposition 4.7(2) to complete the factor of  $\mathcal{S}(\sigma)$  by copies of  $K_{h,h,h}$ .

#### 4.4 Part 3b: $G$ is approximately $\Gamma_3(\lfloor N/3 \rfloor)$

Figure 4 defines the case  $\Gamma_3(\lfloor N/3 \rfloor)$  where sets that are connected with a dotted line are sparse.

We will assume for this part that each vertex is adjacent to at least  $h \lceil \frac{2N}{3h} \rceil + h - 1$  vertices in each of the other pieces of the partition. We also assume that  $G$  is not in the very extreme case. We must deal with the very extreme case separately.

Let  $t \stackrel{\text{def}}{=} h \lfloor N/(3h) \rfloor$ . We will transform the  $\Delta_2$ -approximately  $\Gamma_3(\lfloor N/3 \rfloor)$  by partitioning  $V^{(i)}$ ,  $i = 1, 2, 3$ , into four sets, as follows:  $V^{(i)} = A_1^{(i)} + A_2^{(i)} + A_3^{(i)} + C^{(i)}$ , such that each  $A_j^{(i)}$  has size between  $(1 - \sqrt{\Delta_2})t$  and  $(1 + \sqrt{\Delta_2})t$  and each vertex in  $A_1^{(i)}$  is adjacent to at least  $(1 - \Delta_3)t$  vertices in each set  $A_{j'}^{(i')}$  where  $i' \neq i$  and  $j' \in \{2, 3\}$ . For  $j = 2, 3$ ,  $A_j^{(i)}$  is adjacent to at least  $(1 - \Delta_3)t$  vertices in each set  $A_1^{(i')}$  and  $A_j^{(i')}$ , where  $i' \neq i$ .

Each vertex  $c \in C^{(i)}$  has the property that, for all  $j \in \{1, 2, 3\}$  and distinct  $i', i'' \in \{1, 2, 3\} \setminus \{i\}$ , if  $c$  is adjacent to fewer than  $\Delta_3 t$  vertices in  $A_j^{(i')}$ , then  $c$  is adjacent to at least  $\Delta_3 t$  vertices in  $A_j^{(i'')}$ . Furthermore,  $c$  is adjacent to at least  $(1/2 - \Delta_4)t$  vertices in at least two of  $\{A_1^{(i')}, A_2^{(i')}, A_3^{(i')}\}$  and  $\{A_1^{(i'')}, A_2^{(i'')}, A_3^{(i'')}\}$ .

Without loss of generality, we will assume that both  $|A_2^{(1)}| \geq |A_3^{(1)}|$  and  $|A_2^{(2)}| \geq |A_3^{(2)}|$ .

### Step 1: Ensuring small $A_j^{(i)}$ sets

In each set  $V^{(i)}$ , we construct a set  $Z^{(i)} = Z^{(i)}[1] + Z^{(i)}[2] + Z^{(i)}[3]$  that will contain star-centers.

If  $|A_2^{(3)}| > |A_3^{(3)}|$ , then  $A_2^{(i)}$  is larger than  $A_3^{(i)}$  for  $i = 1, 2, 3$ . Use Lemma 4.4(1) to construct  $\max\{\min\{|A_2^{(i)}| - t, t - |A_3^{(i)}|\}, 0\}$  disjoint copies of  $K_{1,h}$  in the pair<sup>1</sup>  $(A_2^{(i)}, A_3^{(i+1)})$  with centers in  $A_2^{(i)}$ . Place these centers into  $Z^{(i)}[3]$ .

If  $|A_2^{(3)}| < |A_3^{(3)}|$ , we do something similar except that first we use Lemma 4.4(1) to create the appropriate number of stars in  $(A_2^{(1)}, A_3^{(2)})$  and  $(A_2^{(2)}, A_3^{(1)})$  with the centers in  $A_2^{(1)}$  and  $A_2^{(2)}$ , respectively. Place these centers into  $Z^{(1)}[3]$  and  $Z^{(2)}[3]$ , respectively. Then, we apply Lemma 4.4(1) to the pair  $(A_3^{(3)}, A_2^{(2)})$ . (This  $A_2^{(2)}$  is the possibly modified set, with star-centers removed.)

By the conditions on Lemma 4.4(1), we see that each remaining set  $A_j^{(i)}$  is of size at most  $t$ . Now, apply Lemma 4.4(2) to the triple  $(A_1^{(1)}, A_1^{(2)}, A_1^{(3)})$ . For star-centers in  $A_1^{(i)}$ , place  $t - |A_2^{(i)}|$  into  $Z^{(i)}[2]$  and  $t - |A_3^{(i)}|$  into  $Z^{(i)}[3]$ .

### Step 2: Fixing the size of the $A_j^{(i)}$ sets for $j = 1, 2, 3$

We now attempt to “fill up” the sets  $A_j^{(i)}$ . Let  $s_{i,j}$  be the targeted size. There are several cases according to the divisibility of  $N/h$ . Let  $N/h = 6q + r$  where  $0 \leq r < 6$ .

- $r = 0$ :  $s_{i,j} = t$  for  $i = 1, 2, 3$  and  $j = 1, 2, 3$
- $r = 1$ :  $s_{i,j} = t$  for  $i = 1, 2, 3$  and  $j = 1, 3$ ; and  $s_{i,2} = t + h$  for  $i = 1, 2, 3$
- $r = 2$ :  $s_{i,1} = t$  for  $i = 1, 2, 3$ ; and  $s_{i,j} = t + h$  for  $i = 1, 2, 3$  and  $j = 2, 3$
- $r = 3$ :  $s_{i,j} = t$  for  $i = 1, 2, 3$  and  $j = 1, 2, 3$
- $r = 4$ :  $s_{i,1} = t$  for  $i = 1, 2, 3$ ; and  $s_{1,3} = s_{2,3} = s_{3,2} = t$ ; and  $s_{1,2} = s_{2,2} = s_{3,3} = t + h$
- $r = 5$ :  $s_{i,1} = t$  for  $i = 1, 2, 3$ ; and  $s_{i,j} = t + h$  for  $i = 1, 2, 3$  and  $j = 2, 3$

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<sup>1</sup>Arithmetic in the indices is always done modulo 3.

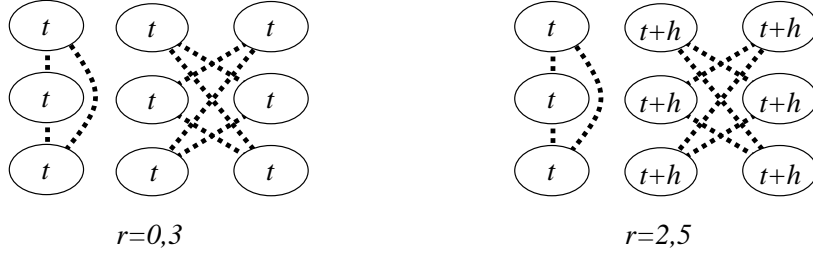


Figure 5:  $N = (6q + r)h$  with  $r = 0, 3$  and  $r = 2, 5$ , respectively.  $t = 2qh + h\lfloor r/3 \rfloor$

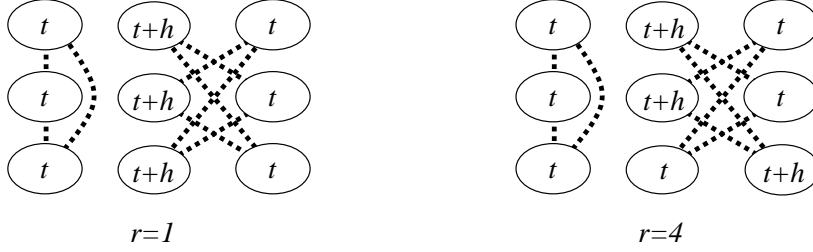


Figure 6:  $N = (6q + r)h$  with  $r = 1$  and  $r = 4$ , respectively.  $t = 2qh + h\lfloor r/3 \rfloor$

The cases of  $r = 0, 3$  and  $r = 2, 5$  are diagrammed in Figure 5 and the cases of  $r = 1$  and  $r = 4$  are diagrammed in Figure 6.

Place vertices of  $Z^{(i)}[j]$  into  $A_j^{(i)}$  for  $i = 1, 2, 3$  and  $j = 1, 2, 3$ . Furthermore, place vertices from  $C^{(i)}$  into  $A_j^{(i)}$  for  $i = 1, 2, 3$  and  $j = 1, 2, 3$ , ensuring that we still have the case that  $|A_j^{(i)}| \leq s_{i,j}$ .

As usual, we call a vertex in  $A_j^{(i)}$  a typical vertex if it was neither in  $C^{(i)}$  nor is either a star-leaf or a star-center. For  $j = 2, 3$ , let  $\mathcal{A}_j = (A_j^{(1)}, A_j^{(2)}, A_j^{(3)})$ . We remove some copies of  $K_{h,h,h}$  from among typical vertices of these sets as follows:

- $r = 1$ : One from  $\mathcal{A}_2$ .
- $r = 2$ : One from each of  $\mathcal{A}_2$  and  $\mathcal{A}_3$ .
- $r = 4$ : One from  $\mathcal{A}_2$ .
- $r = 5$ : Two from  $\mathcal{A}_2$ .

Recalling  $N = (6q + r)h$ , each set is of size  $2qh$  or  $2qh + h$ . Here we note that  $t_f \stackrel{\text{def}}{=} h\lfloor t/(2h) \rfloor = qh$ . Also,  $t_c \stackrel{\text{def}}{=} h\lfloor t/(2h) \rfloor = qh$  if  $r = 0, 1, 2$  and  $t_c = (q + 1)h$  if  $r = 3, 4, 5$ .

### Step 3a: Partitioning the sets ( $r \neq 3$ )

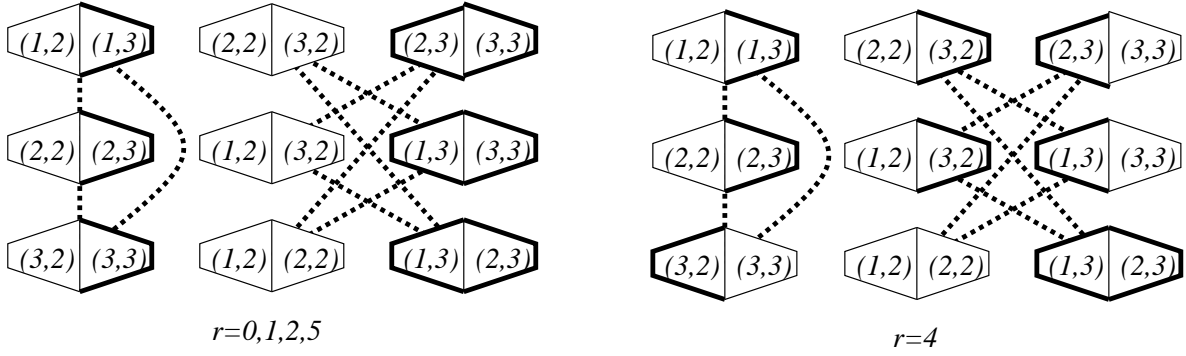


Figure 7: Partitioning the sets. The light outlined half of a set is the piece of size  $t_f$ , the bold outlined half of a set is the piece of size  $t_c$ .

Let  $r \in \{0, 1, 2, 4, 5\}$ . Partition each  $A_1^{(i)}$  set into parts of nearly equal size. Each part of the partition will receive a label  $\sigma \in \{1, 2, 3\} \times \{2, 3\}$ . Now, partition each  $A_j^{(i)}$  as follows:

Each  $A_1^{(i)}$  will be split into two pieces: one of size  $t_f$  and another of size  $t_c$ . Unless both  $r = 4$  and  $i = 3$ , assign the smaller one with label  $(i, 2)$  and the larger with label  $(i, 3)$ . If they are the same size, then assign them arbitrarily. If  $r = 4$  and  $i = 3$ , then assign the one of size  $t_f$  with label  $(3, 3)$  and the one of size  $t_c$  with  $(3, 2)$ .

Each  $A_2^{(i)}$  will be split into two pieces. Unless both  $r = 4$  and  $i \in \{1, 2\}$ , both pieces will be of size  $t_f$  and will be assigned  $(i', 2)$  and  $(i'', 3)$  arbitrarily, where  $\{i, i', i''\} = \{1, 2, 3\}$ . If  $r = 4$  and  $i \in \{1, 2\}$ , the one of size  $t_f$  is labeled  $(3, 2)$  and the one of size  $t_c$ , is labeled  $(3 - i, 2)$ .

Each  $A_3^{(i)}$  will be split into two pieces. Unless both  $r = 4$  and  $i \in \{1, 2\}$ , both pieces will be of size  $t_c$  and will be assigned  $(i', 2)$  and  $(i'', 3)$  arbitrarily, where  $\{i, i', i''\} = \{1, 2, 3\}$ . If  $r = 4$  and  $i \in \{1, 2\}$ , the one of size  $t_f$  is labeled  $(3, 3)$  and one of size  $t_c$  is labeled  $(5 - i, 3)$ .

Figure 7 diagrams the partitioning.

Partitioning the sets at random again ensures that the above can be accomplished so that all of the vertices' neighborhoods maintain roughly the same proportion, as in Part 3a, Step 3.

Now we proceed to Step 4.

### Step 3b: Partitioning the vertices ( $r = 3$ , not the very extreme case)

Let  $r = 3$  (recall  $N = (6q + r)h$ ) and let  $G$  not be in the very extreme case. It may be possible that there are additional stars  $K_{1,h}$  between sparse pairs. If it is possible to create enough such

stars so as to move star-centers into  $Z^{(i)}$ , then we can have at least one of these sets  $A_j^{(i)}$  of size at most  $2qh$ . If we are not able to do this,  $G$  must be in the the very extreme case. Without loss of generality, the set to be made small is either  $A_1^{(1)}$  or  $A_3^{(1)}$ .

- Suppose vertices are removed to make  $|A_1^{(1)}| = 2qh$ . We will make the set  $A_2^{(1)}$  of size  $(2q+2)h$  by adding vertices from the sets  $C^{(1)}$ ,  $Z^{(1)}[2]$  and  $Z^{(1)}[1]$ .
- Suppose vertices are removed to make  $|A_3^{(1)}| = 2qh$ . We will make the set  $A_2^{(1)}$  of size  $(2q+2)h$  by adding vertices from the sets  $C^{(1)}$ ,  $Z^{(1)}[3]$  and  $Z^{(1)}[1]$ .

In each case, if the vertices in  $Z^{(1)}[1]$  that were placed into  $A_2^{(1)}$  were themselves originally in  $A_2^{(1)}$ , then we just treat them as typical vertices again, ignoring the star that was formed. Note that all sets are of size  $(2q+1)h$ , except  $|A_2^{(1)}| = (2q+2)h$  and either  $A_1^{(1)}$  or  $A_3^{(1)}$ , which has size  $2qh$ . If  $A_1^{(1)}$  is the small set, then remove one copy of  $K_{h,h,h}$  in the triple  $(A_3^{(1)}, A_3^{(2)}, A_3^{(3)})$ .

Now we partition each set as follows: Each  $A_1^{(i)}$  will have one piece of size  $qh$  with label  $(1, 3)$ . The other set will have label  $(1, 2)$  size  $(q+1)h$  in the case of  $A_1^{(2)}$  and  $A_1^{(3)}$  and either  $qh$  or  $(q+1)h$  in the case of  $A_1^{(1)}$ . The set  $A_2^{(1)}$  is partitioned into two pieces of size  $(q+1)h$ , one labeled  $(2, 2)$ , the other labeled  $(3, 2)$ . For  $A_2^{(i)}$ ,  $i = 2, 3$ , we have one piece of size  $qh$  and labeled  $(1, 2)$  and the other of size  $(q+1)h$ , labeled  $(5-i, 2)$ . For  $A_3^{(1)}$ , it will have two pieces of size  $qh$ , one labeled  $(2, 3)$ , the other  $(3, 3)$ . Finally, for  $A_3^{(i)}$ ,  $i = 2, 3$ , we have one piece of size  $qh$  with label  $(5-i, 3)$  and the other will have size either  $qh$  or  $(q+1)h$  and label  $(1, 3)$ .

Partitioning the sets at random again ensures that the above can be accomplished so that all of the vertices' neighborhoods maintain roughly the same proportion, as in Part 3a, Step 3.

Now, we can proceed to Step 4.

#### Step 4: Assigning vertices

For any  $\sigma \in \{1, 2, 3\} \times \{2, 3\}$ , we will show that the  $Z^{(i)}$  and  $C^{(i)}$  vertices, in any  $A_j^{(i)}$  can be assigned to one of the two parts of the partition.

For example, consider a vertex  $c$  which had been in  $C^{(1)}$  but is now in  $A_1^{(1)}$ . Then, for either the pair  $(A_2^{(2)}, A_2^{(3)})$  or the pair  $(A_3^{(2)}, A_3^{(3)})$ , the vertex  $c$  is adjacent to at least  $(1/2 - \delta)t$  in one set and at least  $\Delta_3 t$  vertices in the other. If such a pair is  $(A_2^{(2)}, A_2^{(3)})$  then if  $c$  were labeled  $(1, 2)$  exchange it with a typical vertex with label  $(1, 3)$ .

Now, for example, consider a vertex  $c$  which had been in  $C^{(1)}$  but is now in  $A_2^{(1)}$ . It is easy to check that for either the pair  $(A_1^{(2)}, A_2^{(3)})$  or the pair  $(A_1^{(3)}, A_2^{(2)})$ , the vertex  $c$  is adjacent to at least  $(1/2 - \Delta_3)t$  in one set and at least  $\Delta_3 t$  vertices in the other. If such a pair is, say,  $(A_1^{(2)}, A_2^{(3)})$ , and  $c$  is not labeled  $(2, 2)$ , then exchange it for a typical vertex of that label.

Without loss of generality, this takes care of those vertices  $c \in C^{(i)}$ .

Now we consider stars. All star-centers are in sets  $A_2^{(i)}$  or  $A_3^{(i)}$ . Without loss of generality, assume  $z$  is such a center in  $A_2^{(1)}$  and the leaves are in  $V^{(2)}$ . If the leaves are in  $A_1^{(2)}$ , then  $z$  must have been a member of  $A_1^{(1)}$  originally. So,  $z$  and its leaves must have label  $(2, 2)$ . If the leaves are in  $A_2^{(2)}$ , then  $z$  must have been a member of  $A_3^{(1)}$  originally. So,  $z$  and its leaves must have label  $(3, 2)$ . Exchange  $z$  with typical vertices to ensure this.

Finally, we consider typical vertices moved from  $A_2^{(i)} \cup A_3^{(i)}$  to  $A_1^{(i)}$ . Without loss of generality, suppose  $z$  is such a vertex in  $A_1^{(1)}$ . If  $z$  were originally from  $A_2^{(1)}$ , then it is a typical vertex with respect to  $A_2^{(2)}$  and  $A_2^{(3)}$  and  $z$  should receive label  $(1, 2)$ . Otherwise, it is typical with respect to  $A_3^{(2)}$  and  $A_3^{(3)}$  and  $z$  should receive label  $(1, 3)$ .

This completes the verification that all moved vertices can receive at least one label of the  $A_j^{(i)}$  set in which it is placed.

### Step 5: Completing the cover

For any  $\sigma \in \{1, 2, 3\} \times \{2, 3\}$ , let  $\mathcal{S}(\sigma)$  be one of the triples defined above. We can finish as in Part 3a, Step 5.

## 4.5 The very extreme case

Recall the very extreme case:

There are integers  $N, q$  such that  $N = (6q + 3)h$ . There are sets  $A_j^{(i)}$  for  $i, j \in \{1, 2, 3\}$ , with sizes at least  $2qh + 1$ , such that if  $v \in A_j^{(i)}$  then  $v$  is nonadjacent to at most  $3h - 3$  vertices in  $A_{j'}^{(i')}$  whenever the pair  $(A_j^{(i)}, A_{j'}^{(i')})$  corresponds to an edge in the graph  $\Gamma_3$  with respect to the usual correspondence.

In this case, we must raise the minimum degree condition to  $2N/3 + 2h - 1$ . Recalling Part 4, Step 3b, we were able to proceed if we were able to make one of the sets  $A_j^{(i)}$  small by means of creating stars. Each vertex in  $A_2^{(2)}$  is adjacent to at least  $|A_{1,3}| - N/3 + 2h - 1$  vertices in  $A_3^{(1)}$ . Using Lemma 4.4(1), we have that there is a family of  $|A_3^{(1)}| - N/3 + h$  vertex-disjoint stars with centers in  $A_3^{(1)}$ . We move the centers to  $A_2^{(1)}$ . Then we can proceed from Part 4, Step 4.

## 4.6 Proofs of Lemmas

### Proof of Lemma 4.4.

- (1) Let  $\delta_1 = d_1 - h + 1$ . If the stars cannot be created greedily, then there is a set  $S \subset A^{(1)}$  and  $T \subset A^{(2)}$  such that  $|S| \leq \delta_1 - 1$  and  $|T| = |S|h$  and each vertex in  $A^{(1)} \setminus S$  is adjacent to less than  $h - 1$  vertices in  $A^{(2)} \setminus T$ . In this case,

$$(d_1 - |S|)|A^{(2)} \setminus T| \leq e(A^{(1)} \setminus S, A^{(2)} \setminus T) \leq (h - 1)|A^{(1)} \setminus S|.$$

This gives

$$\begin{aligned} |S| &\geq \delta_1 - (h - 1) \frac{|A^{(1)} \setminus S| - |A^{(2)} \setminus T|}{|A^{(2)} \setminus T|} \\ &\geq \delta_1 - (h - 1) \frac{|A^{(1)}| - |A^{(2)}| + (h - 1)|S|}{|A^{(2)}| - h|S|} \\ &\geq \delta_1 - (h - 1) \frac{(h + 1)\epsilon M}{(1 - (h + 1)\epsilon)M}. \end{aligned}$$

If  $\epsilon < (h^2 + h)^{-1}$ , then this gives  $|S| > \delta_1 - 1$ . Since  $|S|$  is an integer,  $|S| \geq \delta_1$ , contradicting the condition we put on  $|S|$ .

- (2) Let  $\delta_i = \max\{0, d_i - h + 1\}$  for  $i = 1, 2, 3$ . If, say,  $\delta_3 = 0$ , then apply part (1) to the pair  $(A^{(2)}, A^{(3)})$  to create  $\delta_2$  vertex-disjoint stars with centers in  $A^{(2)}$ . Let  $Z^{(2)}$  be the set of the centers. Apply part (1) to  $(A^{(1)}, A^{(2)} \setminus Z^{(2)})$  and we can find  $\delta_1$  vertex-disjoint stars with centers in  $A^{(1)}$  if  $2\epsilon < (h^2 + h)^{-1}$ .

So, we may assume that  $\delta_i > 0$  for  $i = 1, 2, 3$ . Note that if it is possible to construct  $\delta_1 + \delta_2$  disjoint copies of  $K_{1,h}$  in  $(A^{(1)}, A^{(2)})$  with centers,  $Z^{(1)} \subset A^{(1)}$ , then we can finish with application of part (1). To see this, apply part (1) to  $(A^{(3)}, A^{(1)} \setminus Z^{(1)})$ , with  $3\epsilon < (h^2 + h)^{-1}$ , creating  $\delta_3$  stars with centers  $Z^{(3)} \in A^{(3)}$ . Then apply part (1) to  $(A^{(2)}, A^{(3)} \setminus Z^{(3)})$  ( $2\epsilon < (h^2 + h)^{-1}$ ). There will be  $\delta_1$  stars remaining in  $(A^{(1)}, A^{(2)})$  which are vertex-disjoint from the rest.

So, we will assume that it is not possible to create  $\delta_1 + \delta_2$  vertex-disjoint copies of  $K_{1,h}$  in  $(A^{(1)}, A^{(2)})$  with centers in  $A^{(1)}$ . That means there is an  $S \subset A^{(1)}$  and a  $T \subset A^{(2)}$  such that  $|S| < \delta_1 + \delta_2$ ,  $|T| = h|S|$  and every vertex in  $A^{(1)} \setminus S$  is adjacent to at most  $h - 1$  vertices in  $A^{(2)} \setminus T$ .

Now apply part (1) to  $(A^{(3)}, A^{(1)} \setminus S)$  to obtain  $\delta_3$  vertex-disjoint copies of  $K_{1,h}$  with centers  $Z^{(3)} \subset A^{(3)}$ . (Here, we need  $3\epsilon < (h^2 + h)^{-1}$ .) Next, apply part (1) to  $(A^{(2)}, A^{(3)} \setminus Z^{(3)})$  to obtain  $\delta_2$  vertex-disjoint copies of  $K_{1,h}$  with centers  $Z^{(2)} \subset A^{(2)}$ . (Here, we need  $2\epsilon < (h^2 + h)^{-1}$ .) Finally, apply part (1) to  $(A^{(1)}, A^{(2)} \setminus (Z^{(2)} \cup T))$  to obtain  $\delta_1$  vertex-disjoint copies of  $K_{1,h}$  with centers  $Z^{(1)} \subset A^{(1)}$ . (Here, we need  $(2h + 2)\epsilon < (h^2 + h)^{-1}$ .) But, because no vertex in  $A^{(1)} \setminus S$  is adjacent to  $h$  vertices in  $A^{(2)} \setminus (Z^{(2)} \cup T)$ , it must be the case that  $Z^{(1)} \subset S$  and our  $\delta_1 + \delta_2 + \delta_3$  copies of  $K_{1,h}$  are, indeed, vertex-disjoint.

□

**Proof of Lemma 4.5.** We can first apply the following theorem of Erdős, Frankl and Rödl [6]:

**Theorem 4.10** *For every  $\epsilon' > 0$  and graph  $F$ , there is a constant  $n_0$  such that for any graph  $G$  of order  $n \geq n_0$ , if  $G$  does not contain  $F$  as a subgraph, then  $G$  contains a set  $E'$  of at most  $\epsilon'n^2$  edges such that  $G \setminus E'$  contains no  $K_r$  with  $r = \chi(F)$ .*

Here,  $F = K_{1,h,h}$  and  $r = 3$ .

So, after removing at most  $\epsilon'(3M)^2$  edges, we have that the number of vertices in each part that are adjacent to at least  $\sqrt{\epsilon}M$  vertices in each of the other two parts is at least  $\left(1 - \frac{18\epsilon'}{\sqrt{\epsilon}-\epsilon}\right)M$ .

So, now  $2\left(1 - \frac{9\epsilon'}{\sqrt{\epsilon}-\epsilon}\right)M \leq |B^{(i)}| \leq 2\left(1 + \frac{\epsilon}{2}\right)M$  and each vertex is adjacent to at least  $\left(1 - \frac{18\epsilon'}{\sqrt{\epsilon}-\epsilon}\right)M$  vertices in each of the other two parts.

Finally, we use a version of a proposition appearing in [19], rephrased below:

**Proposition 4.11** *For a  $\Delta$  small enough, there exists  $\epsilon'' > 0$  such that if  $H$  is a tripartite graph with at least  $2(1 - \epsilon'')t$  vertices in each vertex class and each vertex is nonadjacent to at most  $(1 + \epsilon'')t$  vertices in each of the other classes. Furthermore, let  $H$  contain no triangles. Then, each vertex class is of size at most  $2(1 + \epsilon'')t$  and  $H$  is  $\Delta$ -approximately  $\Theta_{3 \times 2}(t)$ .*

By guaranteeing  $\epsilon'' \gg \epsilon' \gg \epsilon$  and  $\delta = \Delta(\epsilon'') + \epsilon''$ , the lemma follows.  $\square$

**Proof of Lemma 4.6.**

Let  $\epsilon'$  be chosen such that  $\epsilon' \ll \delta$ .

For this lemma, we partition the possibilities according to whether the pairs  $(B^{(i)}, B^{(j)})$  are approximately  $\Theta_{2 \times 2}(t_1)$ . That is, there are two sets of size  $t_1$  which have density less than  $\epsilon'$ . Minimality gives the rest.

In addition, we say that graphs  $\Theta_{2 \times 2}(t_1)$  **coincide** if there are sets  $\tilde{B}^{(i)} \subseteq B^{(i)}$ ,  $\tilde{B}^{(j)} \subseteq B^{(j)}$ ,  $\tilde{B}^{(k)} \subseteq B^{(k)}$ , all of size  $t_1$ , such that both  $(\tilde{B}^{(i)}, \tilde{B}^{(j)})$  and  $(\tilde{B}^{(j)}, \tilde{B}^{(k)})$  have density less than  $\epsilon'$ .

### Case 1: No pair is $\Theta_{2 \times 2}(t_1)$

For each distinct  $i, j, k \in \{1, 2, 3\}$ , partition  $B^{(i)}$  into two pieces,  $B^{(i)}[j]$  and  $B^{(i)}[k]$  with  $|B^{(i)}[j]| = t_j$  and  $|B^{(i)}[k]| = t_k$ . If this partition is done uniformly at random, then with probability approaching 1, each vertex in  $B^{(i)}[k]$  is adjacent to at least  $(1/2 - \epsilon^{1/2})t_k$  vertices in  $B^{(j)}[k]$ . So there exists a partition such that each vertex in  $B^{(i)}$  is adjacent to at least  $(1/2 - \epsilon^{1/2})t_1$  vertices in each of the pieces  $B^{(j)}[k]$ ,  $j, k \neq i$  and such that the pair  $(B^{(2)}[1], B^{(3)}[1])$  fails to contain a subpair with  $\lfloor t_1/2 \rfloor$  vertices in each part and density at most  $\epsilon^{1/3}$ .

The vertices that are reserved will have to be placed in the proper set. For example, if a reserved  $K_{h,h}$  is in the pair  $(B^{(i)}, B^{(j)})$ , then those vertices will need to be in the pair  $(B^{(i)}[k], B^{(j)}[k])$ . So, we exchange vertices in  $B^{(i)}[k]$  for vertices in  $B^{(i)}[j]$  so that reserved vertices are in the proper place. At most  $4(\epsilon + \epsilon)t_1$  vertices are either reserved or moved in each set  $B^{(i)}[j]$ . After such exchanges occur, place the moved vertices into vertex-disjoint copies of  $K_{h,h}$  that lie entirely within the given pairs. This can be done because each vertex not in  $B^{(i)}$  is adjacent to almost half of the vertices in both  $B^{(i)}[j]$  and  $B^{(i)}[k]$ .

Consider what remains of these sets. The number of vertices is still divisible by  $h$  and at most  $8h(\epsilon)t_1$  have been placed into these copies of  $K_{h,h}$ . We look for a perfect  $K_{h,h}$ -factor in each of the pairs  $(B^{(1)}[3], B^{(2)}[3])$ ,  $(B^{(1)}[2], B^{(3)}[2])$  and  $(B^{(2)}[1], B^{(3)}[1])$ . Recall that each of these pairs has minimum degree at least  $(1/2 - \epsilon^{1/2})t_1$ . Utilizing a lemma in [26] – stated as Lemma 4.8 in section 4.2 – we are able to find such a factor unless at least one of those pairs is  $\alpha(\epsilon^{1/2})$ -approximately  $\Theta_{2 \times 2}(t_1/2)$ . (Minimality gives the other sparse pair.)

Lemma 4.9 says that if random selections give a graph that is approximately  $\Theta_{2 \times 2}$ , then the original graph was, too. So, along with Lemma 4.8, it establishes that if, after moving our vertices, we are unable to complete our  $K_{h,h}$ -cover in  $(B_i(k), B_j(k))$  with nontrivial probability, then the pair  $(B_i, B_j)$  is  $\epsilon'$ -approximately  $\Theta_{2 \times 2}(t_1)$ , where  $\epsilon' = \beta(\alpha(\epsilon^{1/2}))$ .

Since none of the pairs is  $\epsilon'$ -approximately  $\Theta_{2 \times 2}(t_1)$ , we can find the required factor of  $(B^{(1)}, B^{(2)}, B^{(3)})$  by copies of  $K_{h,h}$ .

### Case 2: Exactly one pair is $\Theta_{2 \times 2}(t_1)$

Here, we will assume that  $B^{(1)} = \tilde{B}^{(1)} + \hat{B}^{(1)}$  and  $B^{(2)} = \tilde{B}^{(2)} + \hat{B}^{(2)}$ , where  $|\tilde{B}^{(1)}| = |\hat{B}^{(2)}| = t_1$  and  $d(\tilde{B}^{(1)}, \hat{B}^{(2)}), d(\hat{B}^{(1)}, \tilde{B}^{(2)}) \leq \epsilon'$ . A random partition of  $B^{(1)}$  into pieces, with probability approaching 1 as  $t_1$  approaches infinity, will partition  $\tilde{B}^{(1)}$  into two approximately equal pieces. In particular, let the **typical vertices** in  $\tilde{B}^{(1)}$  be those that are nonadjacent to at most  $(\epsilon')^{1/2}t_1$  in  $\hat{B}^{(2)}$ . There are at most  $(\epsilon')^{1/2}t_1$  such vertices. A similar conclusion can be drawn from  $\tilde{B}^{(2)}, \hat{B}^{(1)}$  and  $\hat{B}^{(2)}$ .

In this case, we randomly partition  $B^{(1)}, B^{(2)}$  and  $B^{(3)}$  into the sets  $B^{(i)}[k]$  as proscribed. Exchange the vertices as we have done above and complete both the reserved and exchanged vertices to form copies of  $K_{h,h}$ . This encompasses at most  $8h\epsilon t_1$  vertices. Exchange vertices in  $B^{(1)}[3]$  with vertices in  $B^{(1)}[2]$  and vertices in  $B^{(2)}[3]$  with vertices in  $B^{(2)}[1]$  so that there are exactly  $h\lfloor t_1/(2h) \rfloor$  typical vertices of  $\tilde{B}^{(1)}$  in  $B^{(1)}[3]$  and  $h\lfloor t_1/(2h) \rfloor$  typical vertices of  $\hat{B}^{(2)}$  in  $B^{(2)}[3]$ . Let the rest of the vertices, not matched into a  $K_{h,h}$ , in  $B^{(1)}[3]$  be typical vertices in  $\hat{B}^{(1)}$  and the rest of the vertices in  $B^{(2)}[3]$  be typical in  $\tilde{B}^{(2)}$ . Using Proposition 4.7(1) on each pair of sets of typical vertices in  $(B^{(1)}[3], B^{(2)}[3])$  will easily have a  $K_{h,h}$ -factor. With  $\epsilon'$  small enough, we can guarantee that at most  $(\epsilon')^{1/3}t_1$  vertices in  $(B^{(1)}[2], B^{(3)}[2])$  and  $(B^{(2)}[1], B_3[1])$  were moved. Applying Lemmas 4.8

and 4.9, and the fact that no pair other than  $(B^{(1)}, B^{(2)})$  can be  $\epsilon'$ -approximately  $\Theta_{2 \times 2}(t_1)$ , we conclude that the pairs  $(B^{(1)}[2], B^{(3)}[2])$  and  $(B^{(2)}[1], B_3[1])$  can be completed to  $K_{h,h}$ -factors.

**Case 3: Exactly two pairs are  $\Theta_{2 \times 2}(t_1)$ , which do not coincide**

Let the pairs in question be  $(B^{(1)}, B^{(2)})$  and  $(B^{(2)}, B^{(3)})$ . Let the dense pairs in the subgraph induced by  $(B^{(1)}, B^{(2)})$  be  $(\tilde{B}^{(1)}, \tilde{B}^{(2)})$  and  $(\hat{B}^{(1)}, \hat{B}^{(2)})$ . Let the dense pairs in  $(B^{(2)}, B^{(3)})$  be  $(\mathring{B}^{(2)}, \mathring{B}^{(3)})$  and  $(\ddot{B}^{(2)}, \ddot{B}^{(3)})$ . Moreover, since the pairs fail to coincide, we can conclude that the intersection of the typical vertices of  $\tilde{B}^{(2)}$  with the typical vertices of each of  $\mathring{B}^{(2)}$  and  $\ddot{B}^{(2)}$  is at least  $(\epsilon')^{1/4}t_1$  and similarly for  $\hat{B}^{(2)}$ .

Once again, we randomly partition the vertices in  $B^{(1)}$ ,  $B^{(2)}$  and  $B^{(3)}$  and move vertices so as to ensure that the reserved vertices and the vertices exchanged for them are placed into vertex-disjoint copies of  $K_{h,h}$ . Our concern at this point is the vertices in  $B^{(2)}$ .

Consider the vertices in  $(B^{(1)}[3], B^{(2)}[3])$ . Approximately half are typical vertices of  $\tilde{B}^{(2)}$  and approximately half are typical vertices of  $\hat{B}^{(2)}$ . Take each non-typical vertex in  $B^{(1)}[3]$  and in  $B^{(2)}[3]$ , match them with a copy of  $K_{h,h}$  in the pair  $(B^{(1)}[3], B^{(2)}[3])$  and remove them. Do the same for vertices in  $B^{(2)}[1]$  that are not typical in  $\mathring{B}^{(2)}$  or  $\ddot{B}^{(2)}$  and in  $B^{(3)}[1]$  that are not typical in  $\mathring{B}^{(3)}$  or  $\ddot{B}^{(3)}$ . Remove those copies of  $K_{h,h}$  also.

Observe that there are at least  $\epsilon^{1/4}t_1/4$  vertices in each intersection of  $\tilde{B}^{(2)}$  or  $\hat{B}^{(2)}$  with  $\mathring{B}^{(2)}$  or  $\ddot{B}^{(2)}$  and with  $B^{(2)}[3]$  or  $B^{(2)}[1]$ .

First, move  $a$  vertices from  $\tilde{B}^{(2)} \cap \mathring{B}^{(2)} \cap B^{(2)}[3]$  to  $\tilde{B}^{(2)} \cap \mathring{B}^{(2)} \cap B^{(2)}[1]$  to make  $|\tilde{B}^{(2)} \cap B^{(2)}[3]|$  divisible by  $h$ . Second, move  $a + b$  vertices from  $\hat{B}^{(2)} \cap \mathring{B}^{(2)} \cap B^{(2)}[1]$  to  $\hat{B}^{(2)} \cap \mathring{B}^{(2)} \cap B^{(2)}[3]$  to make  $|\mathring{B}^{(2)} \cap B^{(2)}[1]|$  divisible by  $h$ . Third, move  $a + b + c$  vertices from  $\hat{B}^{(2)} \cap \ddot{B}^{(2)} \cap B^{(2)}[3]$  to  $\hat{B}^{(2)} \cap \ddot{B}^{(2)} \cap B^{(2)}[1]$ . This will make both  $|\hat{B}^{(2)} \cap B^{(2)}[3]|$  and  $|\ddot{B}^{(2)} \cap B^{(2)}[1]|$  divisible by  $h$ .

Here  $a$ ,  $b$  and  $c$  are the remainders of  $|\tilde{B}^{(2)} \cap B^{(2)}[3]|$ ,  $|\mathring{B}^{(2)} \cap B^{(2)}[1]|$  and  $|\hat{B}^{(2)} \cap B^{(2)}[3]|$ , respectively, when each is divided by  $h$ . Observe that both  $|\tilde{B}^{(2)} \cap B^{(2)}[3]| + |\hat{B}^{(2)} \cap B^{(2)}[3]|$  and  $|\mathring{B}^{(2)} \cap B^{(2)}[1]| + |\ddot{B}^{(2)} \cap B^{(2)}[1]|$  are divisible by  $h$ .

Finally, we exchange vertices in  $\tilde{B}^{(1)} \cap B^{(1)}[3]$  with those in  $\tilde{B}^{(1)} \cap B^{(1)}[2]$  so that  $|\tilde{B}^{(1)} \cap B^{(1)}[3]| = |\tilde{B}^{(2)} \cap B^{(2)}[3]|$  and similarly for  $\hat{B}^{(2)}$ . Also, exchange vertices in  $\mathring{B}^{(3)} \cap B^{(3)}[1]$  with those in  $\mathring{B}^{(3)} \cap B^{(3)}[2]$  so that  $|\mathring{B}^{(3)} \cap B^{(3)}[1]| = |\mathring{B}^{(2)} \cap B^{(2)}[1]|$  and similarly for  $\ddot{B}^{(2)}$ .

Then, in  $(\tilde{B}^{(1)} \cap B^{(1)}[3], \tilde{B}^{(2)} \cap B^{(2)}[3])$ , first greedily place each moved vertex into copies of  $K_{h,h}$  and then finish the factor via Proposition 4.7(1). Do the same for  $(\hat{B}^{(1)} \cap B^{(1)}[3], \hat{B}^{(2)} \cap B^{(2)}[3])$ ,  $(\mathring{B}^{(2)} \cap B^{(2)}[1], \mathring{B}^{(3)} \cap B^{(3)}[1])$  and  $(\ddot{B}^{(2)} \cap B^{(2)}[1], \ddot{B}^{(3)} \cap B^{(3)}[1])$ .

Finally, we can complete the factor of  $(B^{(1)}[2], B^{(3)}[2])$  because if it is not possible, Lemmas 4.8 and 4.9 would require  $(B^{(1)}, B^{(3)})$  to be approximately  $\Theta_{2 \times 2}(t_1)$ , excluded by this case.

**Case 4: Three pairs are  $\Theta_{2 \times 2}(t_1)$ , none of which coincide**

Let the dense pairs in  $(B^{(1)}, B^{(2)})$  be  $(\tilde{B}^{(1)}, \tilde{B}^{(2)})$  and  $(\hat{B}^{(1)}, \hat{B}^{(2)})$ . Let the dense pairs in  $(B^{(2)}, B^{(3)})$  be  $(\mathring{B}^{(2)}, \mathring{B}^{(3)})$  and  $(\ddot{B}^{(2)}, \ddot{B}^{(3)})$ . Let the dense pairs in  $(B^{(1)}, B^{(3)})$  be  $(B_{\#}^{(1)}, B_{\#}^{(3)})$  and  $(B_b^{(1)}, B_b^{(3)})$ . Moreover, since the pairs fail to coincide, we can conclude that the intersection of the typical vertices of one set of sparse pairs with the typical vertices of another is at least  $(\epsilon')^{1/4}t^{(1)}$ .

Partition  $B^{(1)}$ ,  $B^{(2)}$  and  $B^{(3)}$  into appropriately-sized sets as before, uniformly at random. The degree conditions hold with high probability as before. Take non-typical vertices and complete them greedily to place them in vertex-disjoint copies of  $K_{h,h}$  within each of the pairs  $(B^{(1)}[3], B^{(2)}[3])$ ,  $(B^{(2)}[1], B^{(3)}[1])$  and  $(B^{(1)}[2], B^{(3)}[2])$ . Remove these copies of  $K_{h,h}$  from the graph.

Let  $M$  be the largest multiple of  $h$  less than or equal to the size of the intersection of what remains of any sparse set (*i.e.*,  $\tilde{B}^{(i)}, \hat{B}^{(i)}, \mathring{B}^{(i)}, \ddot{B}^{(i)}, B_{\#}^{(i)}, B_b^{(i)}$ ) with a set of the form  $B^{(i)}[k]$ .

We can move vertices as in Case 3 by letting  $a = |\tilde{B}^{(2)} \cap B^{(2)}[3]| - M$ ,  $b = |\mathring{B}^{(2)} \cap B^{(2)}[1]| - M$  and  $c = |\hat{B}^{(2)} \cap B^{(2)}[3]| + M - t_3$ , which is also equal to  $t_1 - M - a - b - |\ddot{B}^{(2)} \cap B^{(2)}[1]|$ . We can perform similar operations to guarantee that, among the vertices that remain in the graph, that

$$\begin{aligned} M &= \left| \tilde{B}^{(1)} \cap B^{(1)}[3] \right| = \left| \tilde{B}^{(2)} \cap B^{(2)}[3] \right| = \left| \mathring{B}^{(2)} \cap B^{(2)}[1] \right| = \left| \mathring{B}^{(3)} \cap B^{(3)}[1] \right| \\ &= \left| B_{\#}^{(1)} \cap B^{(1)}[2] \right| = \left| B_{\#}^{(3)} \cap B^{(3)}[2] \right| \end{aligned}$$

The fact that the pairs do not coincide ensures that there are enough vertices to make these moves.

Place the moved vertices into vertex-disjoint copies of  $K_{h,h}$  and finish the factor via Proposition 4.7(1).  $\square$

**Proof of Proposition 4.7.**

- (1) This is found by arbitrarily placing vertices from the same part into clusters of size  $h$ . Construct an auxiliary graph  $G'$  on the clusters where two are adjacent if and only if they form a  $K_{h,h}$  in  $G$ . Each cluster in  $G'$  is adjacent to at least half of the  $M/h$  clusters in the other part. Using König-Hall, we find a matching in  $G'$ , producing a  $K_{h,h}$ -factor.
- (2) The idea is the same as above – place vertices into clusters of size  $h$  – and use the tripartite version of Proposition 1.3 in [20] as a generalization of König-Hall.

$\square$

## 5 Lower bounds

We give a number of constructions which establish the lower bounds. The constructions in [26] of sparse regular bipartite graphs with no  $C_4$ 's lead naturally to the following important proposition, which we state without proof.

**Proposition 5.1** *For each integer  $d \geq 0$ , there exists an  $n_0$  such that, if  $n \geq n_0$ , there exists a balanced tripartite graph,  $Q(n, d)$ , on  $3n$  vertices such that each of the  $\binom{3}{2}$  natural bipartite subgraphs are  $d$ -regular with no  $C_4$  and  $Q(n, d)$  has no  $K_3$ .*

### 5.1 Tight lower bound for $(6h) \mid N$

Recall that if  $G \in \mathcal{G}_3(N)$ ,  $N \geq N_0$  has minimum degree at least  $h \lceil \frac{2N}{3h} \rceil + (h - 1)$  and is not in the very extreme case, then  $G$  has a  $K_{h,h,h}$ -factor. Proposition 5.2 shows that our results are best possible in the case where  $N$  is a multiple of  $6h$  or even  $N$  is a multiple of  $3h$  but the graph is not in the very extreme case.

**Proposition 5.2** *Fix a natural number  $h \geq 2$  and  $N = 3qh$ . If  $q$  is large enough, there exists a  $G_0 \in \mathcal{G}_3(N)$  such that  $\bar{\delta}(G_0) = h \lceil \frac{2N}{3h} \rceil + h - 2 = 2qh + (h - 2)$  and  $G_0$  has no  $K_{h,h,h}$ -factor.*

**Proof.** We will construct 9 sets  $A_j^{(i)}$  with  $i, j \in \{1, 2, 3\}$ . The union  $A_1^{(i)} + A_2^{(i)} + A_3^{(i)}$  defines the  $i^{\text{th}}$  **vertex-class**. Call the triple  $(A_j^{(1)}, A_j^{(2)}, A_j^{(3)})$  the  $j^{\text{th}}$  **column**.

Construct  $G_0$  as follows: For  $i = 1, 2, 3$ , let  $|A_1^{(i)}| = qh - 1$ ,  $|A_2^{(i)}| = qh$  and  $|A_3^{(i)}| = qh + 1$ . Let the graph in column 1 be  $Q(qh - 1, h - 3)$ , the graph in column 2 be  $Q(qh, h - 2)$  and the graph in column 3 be  $Q(qh + 1, h - 1)$ . If two vertices are in different columns and different vertex-classes, then they are adjacent. It is easy to verify that  $\bar{\delta}(G_0) = 2qh + (h - 2)$ . Suppose, by way of contradiction, that  $G_0$  has a  $K_{h,h,h}$ -factor.

Since there are no triangles and no  $C_4$ 's in any column, the intersection of a copy of  $K_{h,h,h}$  with a column is either a star, with all leaves in the same vertex-class, or a set of vertices in the same vertex-class. So, each copy of  $K_{h,h,h}$  have at most  $h$  vertices in column 3. A  $K_{h,h,h}$ -factor has exactly  $3q$  copies of  $K_{h,h,h}$  and so the factor has at most  $3qh$  vertices in column 3. But there are  $3qh + 3$  vertices in column 3, a contradiction.  $\square$

### 5.2 General lower bound for $h \mid N$

Proposition 5.3 gives a more general lower bound for cases when  $N/h$  is not divisible by 3, although it leaves a gap of 1 from the upper bound.

**Proposition 5.3** *Fix a natural number  $h \geq 2$  and  $N = (3q + r)h$  for  $r \in \{0, 1, 2\}$ . If  $q$  is large enough, there exists a  $G_1 \in \mathcal{G}_3(N)$  such that  $\bar{\delta}(G_1) = h \lceil \frac{2N}{3h} \rceil + h - 3 = 2qh + rh + (h - 3)$  and  $G_1$  has no  $K_{h,h,h}$ -factor.*

**Proof.** Define  $G_1$  as follows: For  $i = 1, 2, 3$ , let  $|A_1^{(i)}| = qh + rh - 1$ ,  $|A_2^{(i)}| = qh$  and  $|A_3^{(i)}| = qh + 1$ . Let the graph in column 1 be  $Q(qh + rh - 1, rh + h - 4)$  if  $rh + h - 4 \geq 0$  and empty otherwise, the graph in column 2 be  $Q(qh, h - 3)$  and the graph in column 3 be  $Q(qh + 1, h - 2)$ . If two vertices are in different columns and different vertex-classes, then they are adjacent. It is easy to verify that  $\bar{\delta}(G_1) = 2qh + rh + (h - 3)$ . Suppose, by way of contradiction, that  $G_1$  has a  $K_{h,h,h}$ -factor.

Since there are no triangles and no  $C_4$ 's in any column, the intersection of a copy of  $K_{h,h,h}$  with a column is either a star, with all leaves in the same vertex-class, or a set of vertices in the same vertex-class. So each copy of  $K_{h,h,h}$  has at most  $h + 1$  vertices in column 1,  $h$  vertices in column 2 and at most  $h$  vertices in column 3.

There are three cases for a copy of  $K_{h,h,h}$ . Case 1 has  $h$  vertices in each column. Case 2 has  $h + 1$  vertices in column 1,  $h - 1$  vertices in column 2 and  $h$  vertices in column 3. Case 3 has  $h + 1$  vertices in column 1,  $h$  vertices in column 2 and  $h - 1$  vertices in column 3.

Since a  $K_{h,h,h}$  having  $h$  vertices in column 3 implies the vertices have the same vertex-class, cases 1 and 2 imply that all vertices in column 3 are in the same vertex-class. Consider case 3. Having  $h$  vertices in column 2 means that all are in the same vertex-class. Since  $h + 1$  vertices in column 1 means that they form a star, the remaining  $h - 1$  vertices in column 3 must be in the same vertex-class (the same vertex-class as the center of the star). Hence, every copy of  $K_{h,h,h}$  has all of its column 3 vertices in the same vertex-class. Therefore, the number of copies of  $K_{h,h,h}$  in a factor is at least  $3 \lceil \frac{qh+1}{h} \rceil = 3q + 3$ , a contradiction because the factor has exactly  $3q + r \leq 3q + 2$  copies of  $K_{h,h,h}$ .  $\square$

Note that in the previous proof, column 1 could be  $Q(qh + rh - 1, rh + h - 3)$  and column 2 could be  $Q(qh, h - 2)$  and the argument does not change. Unfortunately, this proof does require that column 3 have degree at most  $h - 2$  between parts.

### 5.3 Lower bound for the very extreme case

Proposition 5.4 gives a graph in the very extreme case that has minimum degree  $2N/3 + h - 2$ , which is greater than that of Proposition 5.3 but is still far from the upper bound of  $2N/3 + 2h - 2$ .

**Proposition 5.4** *Fix a natural number  $h \geq 2$  and  $N = (6q + 3)h$ . If  $q$  is large enough, there exists a  $G_2 \in \mathcal{G}_3(N)$  in the very extreme case such that  $\bar{\delta}(G_2) = h \lceil \frac{2N}{3h} \rceil + h - 2 = (4q + 2)h + h - 2$  and  $G_2$  has no  $K_{h,h,h}$ -factor.*

**Proof.** Construct  $G_2$  as follows: For  $i = 1, 2, 3$  and  $j = 1, 2, 3$ , let  $|A_j^{(i)}| = 2qh + h$ . Let  $(A_1^{(1)}, A_1^{(2)}, A_1^{(3)})$  be  $Q(2qh + h, h - 2)$ . For  $i = 1, 2, 3$ , let each vertex in  $A_1^{(i)}$  be adjacent to any vertex in  $A_{j'}^{(i')}$  whenever  $i' \neq i$  and  $j' \neq 1$ . For  $j = 2, 3$ , let  $(A_j^{(1)}, A_j^{(2)}, A_j^{(3)})$  be a complete tripartite graph and for  $i' \neq i$ , let  $(A_2^{(i)}, A_3^{(i')})$  be a  $(h - 2)$ -regular graph with no  $C_4$ . It is easy to verify that  $\bar{\delta}(G_2) = 2qh + rh + (h - 2)$ . Suppose, by way of contradiction, that  $G_2$  has a  $K_{h,h,h}$ -factor.

Since there are no triangles and no  $C_4$ 's in column 1, the intersection of a copy of  $K_{h,h,h}$  with column 1 is either a star, with all leaves in the same vertex-class, or a set of vertices in the same vertex-class. So, its intersection is at most  $h$  vertices. Since the factor has  $6q + 3$  members and column 1 has  $6qh + 3h$  total vertices, each member of the factor has exactly  $h$  vertices in column 1. As a result, those vertices are in the same vertex-class.

So, the intersection of any member of the  $K_{h,h,h}$ -factor with columns 2 and 3 is a  $K_{h,h}$  with  $h$  vertices in each of two vertex-classes. Suppose this  $K_{h,h}$  has vertices in different columns. Suppose further that it has one vertex in  $A_2^{(1)}$ , then there are at most  $h - 2$  vertices in  $A_3^{(2)}$ . So, there must be at least 2 vertices in  $A_2^{(2)}$  and since there are no  $C_4$ 's, at most 1 vertex in  $A_3^{(1)}$  if  $h \geq 3$ . If  $h = 2$ , then there can be no vertices in  $A_3^{(1)}$ . Regardless, this is a contradiction to the assumption that a  $K_{h,h}$  has vertices in both column 2 and column 3.

So each member of the  $K_{h,h,h}$ -factor either has  $2h$  vertices in column 2 or  $2h$  vertices in column 3. However, there are at most  $\left\lfloor \frac{3(2qh+h)}{2h} \right\rfloor = 3q + 1$  members of the factor with  $2h$  vertices in column 2 and at most  $3q + 1$  members of the factor with  $2h$  vertices in column 3. In either case, there are less than  $h$  vertices in  $A_2^{(1)} \cup A_3^{(1)}$ , a contradiction.  $\square$

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