VERTEX DEGREE SUMS FOR PERFECT MATCHINGS IN 3-UNIFORM HYPERGRAPHS

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Abstract. We determine the minimum degree sum of two adjacent vertices that ensures a perfect matching in a 3-graph without isolated vertex. More precisely, suppose that $H$ is a 3-uniform hypergraph whose order $n$ is sufficiently large and divisible by 3. If $H$ contains no isolated vertex and $\deg(u) + \deg(v) > \frac{5}{9}n^2 - \frac{5}{9}n + 2$ for any two vertices $u$ and $v$ that are contained in some edge of $H$, then $H$ contains a perfect matching. This bound is tight.

1. Introduction

A $k$-uniform hypergraph (in short, $k$-graph) $H$ is a pair $(V, E)$, where $V := V(H)$ is a finite set of vertices and $E := E(H)$ is a family of $k$-element subsets of $V$. A matching of size $s$ in $H$ is a family of $s$ pairwise disjoint edges of $H$. If the matching covers all the vertices of $H$, then we call it a perfect matching. Given a set $S \subseteq V$, the degree $\deg_H(S)$ of $S$ is the number of the edges of $H$ containing $S$. We simply write $\deg(S)$ when $H$ is obvious from the context.

Given integers $\ell < k \leq n$ such that $k$ divides $n$, we define the minimum $\ell$-degree threshold $m_\ell(k, n)$ as the smallest integer $m$ such that every $k$-graph $H$ on $n$ vertices with $\delta_i(H) \geq m$ contains a perfect matching. In recent years the problem of determining $m_\ell(k, n)$ has received much attention, see, e.g., [2, 4, 6, 7, 8, 9, 11, 12, 13, 15, 16, 17, 19, 20, 21]. In particular, Rödl, Ruciński, and Szemerédi [17] determined $m_{k-1}(k, n)$ for all $k \geq 3$ and sufficiently large $n$. Treglown and Zhao [19, 20] determined $m_{\ell}(k, n)$ for all $\ell \geq k/2$ and sufficiently large $n$. For more Dirac-type results on hypergraphs, we refer readers to surveys [14, 25].

In this paper we consider vertex degrees in 3-graphs. Hän, Person and Schacht [4] showed that

$$m_1(3, n) = \left(\frac{5}{9} + o(1)\right)\left(\frac{n}{2}\right). \quad (1)$$

Kühn, Osthus and Treglown [11] and independently Khan [9] later proved that $m_1(3, n) = \left(\frac{n-1}{2}\right) - \left(\frac{2n^3}{2}\right) + 1$ for sufficiently large $n$.

Motivated by the relation between Dirac’s condition and Ore’s condition for Hamilton cycles, Tang and Yan [18] studied the degree sum of two $(k-1)$-sets that guarantees a tight Hamilton cycle in $k$-graphs. Zhang and Lu [22] studied the degree sum of two $(k-1)$-sets that guarantees a perfect matching in $k$-uniform hypergraphs.

It is more natural to consider the degree sum of two vertices that guarantees a perfect matching in hypergraphs. For two distinct vertices $u, v$ in a hypergraph, we call $u, v$ adjacent if there exists an edge containing both of them. The following are three possible ways of defining the minimum degree sum of 3-graphs. Let $\sigma_2(H) = \min\{\deg(u) + \deg(v) : u, v \in V(H)\}$, $\sigma'_2(H) = \min\{\deg(u) + \deg(v) : u$ and $v$ are adjacent$\}$ and $\sigma''_2(H) = \min\{\deg(u) + \deg(v) : u$ and $v$ are not adjacent$\}$.

The parameter $\sigma_2$ is closely related to the Dirac threshold $m_1(3, n)$. We can prove that when $n$ is divisible by 3 and sufficiently large, every 3-graph $H$ on $n$ vertices with $\sigma_2(H) \geq 2((\frac{n-1}{2}) - (\frac{2n^3}{2})) + 1$ contains a perfect matching. Indeed, such $H$ contains at most one vertex $u$ with $\deg(u) \leq (\frac{n-1}{2}) - (\frac{2n^3}{2})$. If $\deg(u) \leq (5/9 - \varepsilon)(\frac{n}{2})$ for some $\varepsilon > 0$, then we choose an edge containing $u$ and find a perfect matching in

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the remaining 3-graph by \([\bullet]\) immediately. Otherwise, \(\delta_1(H) \geq (5/9 - \varepsilon)(n/3)\). We can prove that \(H\) contains a perfect matching by following the same process as in \([\bullet]\). 

On the other hand, no condition on \(\sigma_2^*\) alone guarantees a perfect matching. In fact, let \(H\) be the 3-graph whose edge set consists of all triples that contain a fixed vertex. This \(H\) contains no two disjoint edges even though it satisfies all conditions on \(\sigma_2^*\) (because any two vertices of \(H\) are adjacent).

Therefore we focus on \(\sigma_2'\). More precisely, we determine the largest \(\sigma_2'(H)\) among all 3-graphs \(H\) of order \(n\) without isolated vertex such that \(H\) contains no perfect matching. (Trivially \(H\) contains no perfect matching if it contains an isolated vertex.) Let us define a 3-graph \(H\) without isolated vertex such that \(H\) contains no perfect matching. The following is our main result.

**Theorem 1.** There exists \(n_0 \in \mathbb{N}\) such that the following holds for all integers \(n \geq n_0\) that are divisible by 3. Let \(H\) be a 3-graph of order \(n \geq n_0\) without isolated vertex. If \(\sigma_2'(H) > \sigma_2^*(H^*) = \frac{2}{3}n^2 - \frac{8}{3}n + 2\), then \(H\) contains a perfect matching.

Theorem [\(\bullet\)] actually follows from the following stability result.

**Theorem 2.** There exist \(\varepsilon > 0\) and \(n_0 \in \mathbb{N}\) such that the following holds for all integers \(n \geq n_0\) that are divisible by 3. Suppose that \(H\) is a 3-graph of order \(n \geq n_0\) without isolated vertex and \(\sigma_2'(H) > 2n^2/3 - \varepsilon n^2\), then \(H \subseteq H^*\) or \(H\) contains a perfect matching.

Indeed, if \(\sigma_2'(H) > 2n^2/3 - 8n/3 + 2\), then \(H \not\subseteq H^*\) and by Theorem [\(\bullet\)] \(H\) contains a perfect matching. Furthermore, Theorem [\(\bullet\)] implies that \(H^*\) is the unique extremal 3-graph for Theorem [\(\bullet\)] because all proper subgraphs \(H\) of \(H^*\) satisfy \(\sigma_2'(H) < \sigma_2'(H^*)\).

This paper is organized as follows. In Section 2, we provide preliminary results and an outline of our proof. We prove an important lemma in Section 3 and we complete the proof of Theorem [\(\bullet\)] in Section 4. Section 5 contains concluding remarks and open problems.

**Notation:** Given vertices \(v_1, \ldots, v_t\), we often write \(v_1 \cdots v_t\) for \(\{v_1, \ldots, v_t\}\). The neighborhood \(N(u, v)\) is the set of the vertices \(w\) such that \(uwv \in E(H)\). Let \(V_1, V_2, V_3\) be three vertex subsets of \(V(H)\), we say that an edge \(e \in E(H)\) is of type \(V_1V_2V_3\) if \(e = \{v_1, v_2, v_3\}\) such that \(v_1 \in V_1, v_2 \in V_2\) and \(v_3 \in V_3\).

Given a vertex \(v \in V(H)\) and a set \(A \subseteq V(H)\), we define the link \(L_v(A)\) to be the set of all pairs \(uw\) such that \(u, w \in A\) and \(uw \in E(H)\). When \(A\) and \(B\) are two disjoint sets of \(V(H)\), we define \(L_v(A, B)\) as the set of all pairs \(uw\) such that \(u \in A, w \in B\) and \(uw \in E(H)\).

We write \(0 < a_1 \ll a_2 \ll a_3\) if we can choose the constants \(a_1, a_2, a_3\) from right to left. More precisely there are increasing functions \(f\) and \(g\) such that given \(a_3\), whenever we choose some \(a_2 \leq f(a_3)\) and \(a_1 \leq g(a_2)\), all calculations needed in our proof are valid.

2. PRELIMINARIES AND PROOF OUTLINE

We will need small constants

\[
0 < \varepsilon \ll \eta \ll \gamma \ll \gamma' \ll \rho \ll \tau \ll 1.
\]

Suppose \(H\) is a 3-graph such that \(\sigma_2'(H) > 2n^2/3 - \varepsilon n^2\). Let \(W = \{v \in V(H) : \deg(v) \leq n^2/3 - \varepsilon n^2/2\}\), \(U = V \setminus W\). If \(W = \emptyset\), then [\(\bullet\)] implies that \(H\) contains a perfect matching. We thus assume that \(|W| \geq 1\). Any two vertices of \(W\) are not adjacent – otherwise \(\sigma_2'(H) \leq 2n^2/3 - \varepsilon n^2\), a contradiction. If \(|W| \geq n/3 + 1\), then \(H \subseteq H^*\) and we are done. We thus assume \(|W| \leq n/3\) for the rest of the proof.

Our proof will use the following claim.

1In fact, due to the absorbing method, we only need to verify the extremal case.
Claim 3. If $|W| \geq n/4$, then every vertex of $U$ is adjacent to some vertex of $W$.

Proof. To the contrary, assume that some vertex $u_0 \in U$ is not adjacent to any vertex in $W$. Then we have $\deg(u_0) \leq \binom{|U| - 1}{2} = \binom{n - |W| - 1}{2}$. Since $|W| \geq n/4$ and $n$ is sufficiently large,

$$\deg(u_0) \leq \left( \frac{n - n/4 - 1}{2} \right) = \frac{9}{32}n^2 - \frac{9}{8}n + 1 < \frac{n^2}{3} - \frac{\varepsilon}{2}n^2,$$

which contradicts the definition of $U$. \qed

By Claim 3, when $|W| \geq \frac{n}{4}$, we have $\deg(u) \geq (2n^2/3 - \varepsilon n^2) - \binom{|W|}{2}$ for every $u \in U$. This is stronger than the bound given by the definition of $U$ because

$$\left( \frac{2}{3}n^2 - \varepsilon n^2 \right) - \binom{|W|}{2} \geq \left( \frac{2}{3}n^2 - \varepsilon n^2 \right) - \binom{n - n/4}{2} > \frac{n^2}{3} - \frac{\varepsilon}{2}n^2.$$

Our proof consists of two steps.

Step 1. We prove that $H$ contains a matching that covers all the vertices of $W$.

Lemma 4. There exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose that $H$ is a 3-graph of order $n \geq n_0$ without isolated vertex and $\sigma_2^*(H) > 2n^2/3 - \varepsilon n^2$. Let $W = \{v \in V(H) : \deg(v) \leq n^2/3 - \varepsilon n^2/2\}$. If $|W| \leq n/3$, then $H$ contains a matching that covers every vertex of $W$.

Our approach towards Lemma 4 begins by considering a largest matching $M$ such that every edge of $M$ contains one vertex from $W$ and suppose $|M| < |W|$. If $|W| \leq (1/3 - \gamma)n$, then we choose two adjacent vertices, one from $W$ and the other from $V \setminus W$ to derive a contradiction with $\sigma_2^*(H)$. If $n/3 \geq |W| > (1/3 - \gamma)n$, we use three unmatched vertices, one from $W$ and two from $V \setminus W$ to derive a contradiction.

Step 2. We show that $H$ contains a perfect matching.

Because of Lemma 4, we begin by considering a largest matching $M$ such that $M$ covers every vertex of $W$ and suppose that $|M| < n/3$. After choosing three vertices from $V \setminus V(M)$, we distinguish the cases when $|M| \leq n/3 - \eta n$ and when $|M| > n/3 - \eta n$ and derive a contradiction by comparing upper and lower bounds for the degree sum of these three vertices. When $|M| > n/3 - \eta n$, we need to apply Lemma 3.

In Step 2 we need three simple extremal results. The first lemma is Observation 1.8 of Aharoni and Howard [1]. A $k$-graph $H$ is called $k$-partite if $V(H)$ can be partitioned into $V_1, \ldots, V_k$, such that each edge of $H$ meets every $V_i$ in precisely one vertex. If all parts are of the same size $n$, we call $H$ $n$-balanced.

Lemma 5. Let $F$ be the edge set of an $n$-balanced $k$-partite $k$-graph. If $F$ does not contain $s$ disjoint edges, then $|F| \leq (s - 1)n^{k-1}$.

The bound in the following lemma is tight because we may let $G_1$ be the empty graph and $G_2 = G_3 = K_n$.

Lemma 6. Given two sets $A \subset V$ such that $|A| = 3$ and $|V| = n \geq 4$, let $G_1, G_2, G_3$ be three graphs on $V$ such that no edge of $G_1$ is disjoint from an edge of $G_2$ or $G_3$. Then $\sum_{i=1}^{3} \sum_{v \in A} \deg_{G_i}(v) \leq 6(n - 1)$.

Proof. Assume $A = \{u_1, u_2, u_3\}$ and let $b = n - 3 \geq 1$. We need to show that $\sum_{i=1}^{3} \sum_{j=1}^{3} \deg_{G_i}(u_j) \leq 6b + 12$.

Let $\ell_i$ denote the number of the vertices in $A$ of degree at least $3$ in $G_i$. We distinguish the following two cases:

Case 1: $\ell_1 \geq 1$.

If $\ell_1 \geq 2$, say, $\deg_{G_1}(u_j) \geq 3$ for $j = 1, 2$, then $E(G_1) \subseteq \{u_1u_2\}$ for $i = 2, 3$ – otherwise we can find two disjoint edges, one from $G_1$ and the other from $G_2$ or $G_3$. Therefore, $\sum_{j=1}^{3} \deg_{G_1}(u_j) \leq 2$ for $i = 2, 3$.

Moreover, $\sum_{j=1}^{3} \deg_{G_1}(u_j) \leq 3b + 6$. We have $\sum_{j=1}^{3} \deg_{G_1}(u_j) \leq 3b + 10 < 6b + 12$.

If $\ell_1 = 1$, say, $\deg_{G_1}(u_1) \geq 3$, then $G_i$ is a star centered at $u_1$ for $i = 2, 3$ – otherwise one edge of $G_1$ must be disjoint from one edge of $G_2$ or $G_3$. In this case we have $\sum_{j=1}^{3} \deg_{G_1}(u_j) \leq b + 2 + 4$ and $\sum_{j=1}^{3} \deg_{G_1}(u_j) \leq b + 4$ for $i = 2, 3$. Therefore, $\sum_{i=1}^{3} \sum_{j=1}^{3} \deg_{G_i}(u_j) \leq 3b + 14 < 6b + 12$ as $b \geq 1$.

Case 2: $\ell_1 = 0$.

If $\ell_1 = 3$ for some $i \in \{2, 3\}$, then $E(G_1) = \emptyset$. In this case $\sum_{i=1}^{3} \sum_{j=1}^{3} \deg_{G_i}(u_j) \leq 2(3b + 6) \leq 6b + 12$.  

Lemma 7. Given two sets $A \subseteq V$ such that $|A| = 3$ and $|V| = n \geq 5$, let $G_1, G_2, G_3$ be three graphs on $V$ such that no edge of $G_i$ is disjoint from an edge from $G_j$ for any $i \neq j$. Then $\sum_{i=1}^{3} \sum_{u \in A} \deg_{G_i}(v) \leq 3(n+1)$.

Proof. Assume $A = \{u_1, u_2, u_3\}$ and let $b = n-3 \geq 2$. We need to show that $\sum_{i=1}^{3} \sum_{j=1}^{3} \deg_{G_i}(u_j) \leq 3b+12$.

Let $\ell_i$ denote the number of the vertices in $A$ of degree at least 3 in $G_i$. We distinguish the following two cases:

Case 1: $\ell_i \geq 1$ for some $i \in [3]$.

Without loss of generality, $\ell_1 \geq 1$ and $\deg_{G_1}(u_1) \geq 3$. If $\deg_{G_2}(u_2) \geq 3$ or $\deg_{G_3}(u_3) \geq 3$, say, $\deg_{G_3}(u_3) \geq 3$, then $E(G_i) \subseteq \{u_1u_2\}$ for $i = 2, 3$ – otherwise we can find two disjoint edges $e_1$ and $e_2$ from two different graphs $G_1, G_2, G_3$. In this case $\sum_{j=1}^{3} \deg_{G_i}(u_j) \leq 3b+6$ and $\sum_{j=1}^{3} \deg_{G_i}(u_j) \leq 2$ for $i = 2, 3$, which implies that $\sum_{i=1}^{3} \sum_{j=1}^{3} \deg_{G_i}(u_j) \leq 3b+10$.

Assume $\deg_{G_2}(u_2) \leq 2$ for $j = 2, 3$. We know that $G_i, i = 2, 3$ is a star centered at $u_1$ – otherwise one edge of $G_1$ must be disjoint from one edge of $G_1$, $i \in \{2, 3\}$. If $\deg_{G_2}(u_1) \geq 3$ or $\deg_{G_3}(u_1) \geq 3$, then $G_1$ is also a star centered at $u_1$. In this case $\sum_{j=1}^{3} \deg_{G_i}(u_j) \leq b+4$ for $i \in [3]$, so $\sum_{i=1}^{3} \sum_{j=1}^{3} \deg_{G_i}(u_j) \leq 3b+12$.

Otherwise $\deg_{G_i}(u_1) \leq 2$ for $i = 2, 3$, hence $\sum_{j=1}^{3} \deg_{G_i}(u_j) \leq 4$ for $i = 2, 3$. Since $\sum_{j=1}^{3} \deg_{G_i}(u_j) \leq b+6$, we have $\sum_{i=1}^{3} \sum_{j=1}^{3} \deg_{G_i}(u_j) \leq b+14 \leq 3b+12$.

Case 2: $\ell_i = 0$ for $i \in [3]$.

In this case $\sum_{i=1}^{3} \deg_{G_i}(u_j) \leq 6$ for $i = 1, 2, 3$. Hence $\sum_{i=1}^{3} \sum_{j=1}^{3} \deg_{G_i}(u_j) \leq 18 \leq 3b+12$ as $b \geq 2$. □

3. PROOF OF LEMMA 4

Choose a largest matching of $H$, denoted by $M$, such that every edge of $M$ is of type $UUW$. To the contrary, assume that $|M| \leq |W| - 1$. Let $U_1 = V(M) \cap U$, $U_2 = U \setminus U_1$, $W_1 = V(M) \cap W$, and $W_2 = W \setminus W_1$. Then $|U_1| = 2|M|$, and $|U_2| = n - |W| - 2|M|$. We distinguish the following two cases.

Case 1: $0 < |W| \leq \left(\frac{1}{2} - \gamma\right)n$.

We further distinguish the following two sub-cases:

Case 1.1: $v_0 \in W_2$ is adjacent to a vertex $u_0 \in U_2$.

Let $M' = \{e \in M : \exists u' \in e, |N(v_0, u') \cap U_2| \geq 3\}$. Assume $\{u_1, u_2, v_1\} \in M'$ such that $u_1, u_2 \in U_1$, $v_1 \in W_1$, and $|N(v_0, u_1) \cap U_2| \geq 3$. We claim that

$$N(u_0, v_1) \cap (U_2 \cup \{u_2\}) = \emptyset. \tag{2}$$

Indeed, if $\{u_0, v_1, u_3\} \in E(H)$ for some $u_3 \in U_2$, then we can find $u_4 \in U_2 \setminus \{u_0, u_3\}$ such that $\{v_0, u_1, u_4\} \in E(H)$. Replacing $\{u_1, u_2, v_1\}$ by $\{u_0, v_1, u_3\}$ and $\{v_0, u_1, u_4\}$ gives a larger matching than $M$, a contradiction. The case when $\{u_0, v_1, u_3\} \in E(H)$ is similar.

By the definition of $M'$, there are at most $2(|U_1| - 2|M'|)$ edges containing $v_0$ with one vertex in $U_1 \setminus V(M')$ and one vertex in $U_2$. This implies that

$$\deg(v_0) \leq \binom{|U_1|}{2} + 2|M'||U_2| + 2(|U_1| - 2|M'|) = \binom{|U_1|}{2} + 2|U_1| + |M'||(2|U_2| - 4).$$
By [2], there are at most $|U_1||W_1| - |M'|$ edges consisting of $u_0$, one vertex in $U_1$, and one vertex in $W_1$, and at most $(|U_2| - 1)(|W_1| - |M'|)$ edges consisting of $u_0$, one vertex in $U_2$, and one vertex in $W_1$. Therefore,

$$\deg(u_0) \leq \left(\frac{|U| - 1}{2}\right) + |U_1||W_2| + |U_1||W_1| - |M'| + (|U_2| - 1)(|W_1| - |M'|)$$

$$= \left(\frac{|U| - 1}{2}\right) + |U_1||W| + (|U_2| - 1)|W_1| - |U_2||M'|,$$

and consequently,

$$\deg(v_0) + \deg(u_0) \leq \left(\frac{|U_1|}{2}\right) + 2|U_1| + \left(\frac{|U| - 1}{2}\right) + |U_1||W| + (|U_2| - 1)|W_1| + |M'|(|U_2| - 4).$$

Since $|W| \leq \left(\frac{1}{3} - \gamma\right)n$, we have $|U_2| > 3\gamma n > 4$. As $|M'| \leq |M| = |W_1| = \frac{|U_1|}{2}$, it follows that

$$\deg(v_0) + \deg(u_0) \leq \left(\frac{|U_1|}{2}\right) + 2|U_1| + \left(\frac{|U| - 1}{2}\right) + |U_1||W| + (|U_2| - 1)\frac{|U_1|}{2} + \frac{|U_1|}{2}(|U_2| - 4)$$

$$= \left(\left(\frac{|U|}{2}\right) - \left(\frac{|U_2|}{2}\right)\right) + \left(\frac{|U| - 1}{2}\right) + \left(|W| - \frac{1}{2}\right)|U_1|$$

$$= (|U| - 1)^2 - \left(\frac{|U_2|}{2}\right) + (2|W| - 1)|M|.$$ 

Since $|M| \leq |W| - 1$ and $|U_2| \geq n - 3|W| + 2$, we derive that

$$\deg(v_0) + \deg(u_0) \leq (n - |W| - 1)^2 - \left(\frac{n - 3|W| + 2}{2}\right) + (2|W| - 1)(|W| - 1)$$

$$= \frac{2}{3}n^2 - \frac{7}{3}n + \frac{73}{24} - \frac{3}{2}\left(\frac{n}{3} + \frac{7}{6} - |W|\right)^2.$$ 

Since $|W| \leq \left(\frac{1}{3} - \gamma\right)n$, $0 < \varepsilon \ll \gamma$ and $n$ is sufficiently large, we have

$$\deg(v_0) + \deg(u_0) \leq \frac{2}{3}n^2 - \frac{7}{3}n + \frac{73}{24} - \frac{3}{2}\left(\gamma n + \frac{7}{6}\right)^2 < \frac{2}{3}n^2 - \varepsilon n^2.$$ 

This contradicts our assumption on $\sigma'_2(H)$ because $v_0$ and $u_0$ are adjacent.

**Case 1.2:** No vertex in $W_2$ is adjacent to any vertex in $U_2$.

Fix $v_0 \in W_2$. Since $v_0$ is not adjacent to any vertex in $U_2$, we have $\deg(v_0) \leq \left(\frac{|U_1|}{2}\right) = \left(\frac{2|M|}{2}\right)$. Since $v_0$ is not an isolated vertex, there exists a vertex $u_1 \in U_1$ that is adjacent to $v_0$. By the assumption, $H$ contains no edge containing $u_1$ with one vertex in $U_2$, one vertex in $W_2$. Thus $\deg(u_1) \leq \left(\frac{|U| - 1}{2}\right) + (|U| - 1)|W| - |U_2||W_2|$. Since $|M| \leq |W| - 1$ and $|U| = n - |W|$, it follows that

$$\deg(v_0) + \deg(u_1) \leq \frac{2(|U| - 1)}{2} + \left(\frac{|U| - 1}{2}\right) + (|U| - 1)|W| - (n - 3|W| + 2)$$

$$= \frac{3}{2}\left(|W| - \frac{1}{2}\right)^2 + \frac{1}{2}n^2 - \frac{5}{2}n + \frac{13}{8}.$$ 

Furthermore, since $|W| \leq \left(\frac{1}{3} - \gamma\right)n$ and $0 < \varepsilon \ll \gamma$, we derive that

$$\deg(v_0) + \deg(u_1) \leq \frac{2}{3}\left(n - \gamma n - \frac{1}{2}\right)^2 + \frac{1}{2}n^2 - \frac{5}{2}n + \frac{13}{8} = \left(\frac{2}{3} - \gamma + \frac{3}{2}\gamma^2\right)n^2 - \left(3 - \frac{3}{2}\gamma\right)n + 2$$

$$< \frac{2}{3}n^2 - \varepsilon n^2,$$

contradicting our assumption on $\sigma'_2(H)$.

**Case 2:** $|W| > \left(\frac{1}{3} - \gamma\right)n$.

**Claim 8.** $|M| \geq n/3 - \gamma'n$. 


Proof. To the contrary, assume that $|M| < n/3 - \gamma'n$. Fix $v_0 \in W_2$. Then $\deg(v_0) \leq \left(\frac{|U|}{2}\right) - \left(\frac{|U_2|}{2}\right)$ because there is no edge of type $U_2U_2W_2$. Suppose $u \in U$ is adjacent to $v_0$. Trivially $\deg(u) \leq \left(\frac{|U|}{2}\right) + \left(\frac{|U_1|}{2}\right) = (n-1)(|U| - 1) - \left(\frac{|U_2|}{2}\right)$. Thus
\[
\deg(v_0) + \deg(u) \leq \left(\frac{|U| - 1}{2}\right) + (|U| - 1)|W| + \left(\frac{|U_1|}{2}\right) - \left(\frac{|U_2|}{2}\right) = (n-1)(|U| - 1) - \left(\frac{|U_2|}{2}\right).
\]
Our assumptions imply that $|U| \leq 2n/3 + \gamma n$ and $|U_2| \geq 2\gamma' n$. As a result,
\[
\deg(v_0) + \deg(u) \leq (n-1)\left(\frac{2n}{3} + \gamma n - 1\right) - \left(\frac{2\gamma' n}{2}\right) < \frac{2}{3}n^2 - \varepsilon n^2,
\]
because $\varepsilon \ll \gamma \ll \gamma'$ and $n$ is sufficiently large. This contradicts our assumption on $\sigma_2^2(H)$. \hfill \Box

Fix $u_1 \neq u_2 \in U_2$ and $v_0 \in W_2$. Trivially $\deg(w) \leq \left(\frac{|U|}{2}\right)$ for any vertex $w \in W$ and $\deg(u) \leq \left(\frac{|U|}{2}\right) + \left|W\right|(|U| - 1)$ for any vertex $u \in U$. Furthermore, for any two distinct edges $e_1, e_2 \in M$, we observe that at least one triple of type $UUV$ with one vertex from each of $e_1$ and $e_2$ and one vertex from $\{u_1, u_2, v_0\}$ is not an edge otherwise there is a matching $M_4$ of size three on $e_1 \cup e_2 \cup \{u_1, u_2, v_0\}$ and $M_3 \cup M \setminus \{e_1, e_2\}$ is thus a matching larger than $M$. By Claim 8, $|M| \geq n/3 - \gamma'n$. Thus,
\[
\deg(u_1) + \deg(u_2) + \deg(v_0) \leq 2\left(\left(\frac{|U| - 1}{2}\right) + |W||U| - 1\right) + \left(\frac{|U|}{2}\right) - \left(\frac{n/3 - \gamma'n}{2}\right).
\]
On the other hand, since $|W| > (\frac{4}{3} - \gamma)n \geq n/4$, Claim 2 implies that $u_i$ is adjacent to some vertex in $W$ for $i = 1, 2$. We know that $v_0$ is adjacent to some vertex in $U$. Therefore, $\deg(u_i) > 2(n^2/3 - \varepsilon n^2) - \left(\frac{|U|}{2}\right)$ for $i = 1, 2$, and $\deg(v_0) > 2(n^2/3 - \varepsilon n^2) - \left(\frac{|U|}{2}\right)$ for $i = 1, 2, 3$. It follows that
\[
\deg(u_1) + \deg(u_2) + \deg(v_0) > 3\left(\frac{2n^2}{3} - \varepsilon n^2\right) - 2\left(\frac{|U|}{2}\right) - \left(\frac{|U|}{2}\right) - |W||U| - 1).
\]
The upper and lower bounds for $\deg(u_1) + \deg(u_2) + \deg(v_0)$ together imply that
\[
3\left(\left(\frac{|U| - 1}{2}\right) + |W||U| - 1\right) + \left(\frac{|U|}{2}\right) - \left(\frac{n/3 - \gamma'n}{2}\right) > 3\left(\frac{2n^2}{3} - \varepsilon n^2\right),
\]
or
\[
(|U| - 1)(n-1) - \frac{1}{3}\left(\frac{n/3 - \gamma'n}{2}\right) > \frac{2n^2}{3} - \varepsilon n^2,
\]
which is impossible because $|U| \leq 2n/3 + \gamma n$, $0 < \varepsilon \ll \gamma \ll \gamma' \ll 1$ and $n$ is sufficiently large. This completes the proof of Lemma 4.

4. Proof of Theorem 2

Choose a matching $M$ such that (i) $M$ covers all the vertices of $W$; (ii) subject to (i), $|M|$ is the largest. Lemma 4 implies that such a matching exists. Let $M_1 = \{e \in M : e \cap W \neq 0\}$, $M_2 = M \setminus M_1$, and $U_3 = V(H) \setminus V(M)$. We have $|M_1| = |W|$, $|M_2| = |M| - |W|$, $|U_3| = n - 3|M|$. Suppose to the contrary, that $|M| \leq n/3 - 1$. Fix three vertices $u_1, u_2, u_3$ of $U_3$. We distinguish the following two cases.

Case 1: $|M| \leq n/3 - \eta m$.

Trivially there are at most $3|M|$ edges in $H$ containing $u_i$ and two vertices from the same edge of $M$ for $i = 1, 2, 3$. For any distinct $e_1, e_2$ from $M$, we claim that
\[
\sum_{i=1}^{3} |L_{u_i}(e_1, e_2)| \leq 18.
\]
Indeed, let $H_1$ be a 3-partite subgraph of $H$ induced on three parts $e_1, e_2$, and $\{u_1, u_2, u_3\}$. We observe that $H_1$ does not contain a perfect matching - otherwise, letting $M_1$ be a perfect matching of $H_1$, $(M \setminus \{e_1, e_2\}) \cup M_1$ is a larger matching than $M$, a contradiction. Apply Lemma 5 with $n = k = s = 3$, we obtain that $|E(H_1)| \leq 18$. Therefore $\sum_{i=1}^{3} |L_{u_i}(e_1, e_2)| \leq 18$. 

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For any $e \in M_1$, we claim that
\[ \sum_{i=1}^{3} |L_{u_i}(e, U_3)| \leq 6(|U_3| - 1). \]
Indeed, assume $e = \{v_1, v_2, v_3\} \in M_1$ with $v_1 \in W$. Apply Lemma 6 with $A = \{u_1, u_2, u_3\}$, $V = U_3$, and $G_i = (U_3, L_{v_i}(U_3))$ for $i = 1, 2, 3$. Since $|M| \leq n/3 - 4$, we have $|B| = |U_3| - 3 \geq 2$. By the maximality of $M$, no edge of $G_1$ is disjoint from an edge of $G_2$ or $G_3$. By Lemma 6, $\sum_{i=1}^{3} \sum_{j=1}^{3} \deg_{G_i}(u_j) \leq 6(|U_3| - 1)$.
Hence $\sum_{i=1}^{3} |L_{u_i}(e, U_3)| = \sum_{i=1}^{3} \sum_{j=1}^{3} \deg_{G_i}(u_j) \leq 6(|U_3| - 1).

Similarly, for any $e \in M_2$, we can apply Lemma 7 to obtain that
\[ \sum_{i=1}^{3} |L_{u_i}(e, U_3)| \leq 3(|U_3| + 1). \]

Putting these bounds together gives
\[
\sum_{i=1}^{3} \deg(u_i) \leq 18 \left(\frac{|M|}{2}\right) + 9|M| + 3 \sum_{i=1}^{3} |L_{u_i}(V(M_1), U_3)| + 3 \sum_{i=1}^{3} |L_{u_i}(V(M_2), U_3)| \\
\leq 18 \left(\frac{|M|}{2}\right) + 9|M| + 6|M_1||U_3| - 1 + 3|M_2||U_3| + 1).
\]

Since $|M_1| = |W|$, $|M_2| = |M| - |W|$, $|U_3| = n - 3|M|$, we derive that
\[
\sum_{i=1}^{3} \deg(u_i) \leq 18 \left(\frac{|M|}{2}\right) + 9|M| + 6|W|(n - 3|M| - 1) + 3(|M| - |W|)(n - 3|M| + 1) \\
= (3n - 9|W| + 3)|M| + 3|W|n - 9|W|.
\]

Furthermore, $3n - 9|W| + 3 > 0$ and $|M| \leq n/3 - \eta n$ implies that
\[
\sum_{i=1}^{3} \deg(u_i) \leq (9\eta n - 9)|W| + (1 - 3\eta) n^2 + (1 - 3\eta) n.
\]

If $|W| \leq n/4$, from (3), we have
\[
\sum_{i=1}^{3} \deg(u_i) \leq (9\eta n - 9) \frac{n}{4} + (1 - 3\eta) n^2 + (1 - 3\eta) n = \left(1 - \frac{3}{4} \eta\right) n^2 - \left(3\eta + \frac{5}{4}\right) n,
\]
which contradicts the condition $\sum_{i=1}^{3} \deg(u_i) \geq 3 \left(\frac{n^2}{3} - \varepsilon n^2\right)$ because $u_i \in U_3$ for $i \in [3]$ and $\varepsilon \ll \eta$.

If $|W| > n/4$, Claim 3 implies that $u_i$ is adjacent to one vertex of $W$, $i = 1, 2, 3$. Furthermore, $\deg(w) \leq \left(\frac{|U|}{2}\right)$ for $w \in W$. So
\[
\sum_{i=1}^{3} \deg(u_i) > 3 \left(\frac{2n^2}{3} - \varepsilon n^2 - \left(\frac{|U|}{2}\right)\right) = 3 \left(\frac{2n^2}{3} - \varepsilon n^2 - \left(\frac{n - |W|}{2}\right)\right).
\]

The upper and lower bounds for $\sum_{i=1}^{3} \deg(u_i)$ together imply that
\[
(9\eta n - 9) |W| + (1 - 3\eta) n^2 + (1 - 3\eta) n + 3 \left(\frac{n - |W|}{2}\right) > 3 \left(\frac{2n^2}{3} - \varepsilon n^2\right),
\]
which is a contradiction because $|W| > n/4$, $0 < \varepsilon \ll \eta \ll 1$ and $n$ is sufficiently large.

**Case 2:** $|M| > n/3 - \eta n$.

If $|M| = n/3 - 1$, then $|U_3| = 3$ and we can not apply Lemmas 6 and 7. In fact, whenever $|M| > n/3 - \eta n$, Lemma 6 suffices for our proof.
Let \( W' = \{ v \in W : \deg(v) \leq (5/18 + \tau)n^2 \} \). Let \( M' \) be the sub-matching of \( M \) covering every vertex of \( W' \). If \( |W'| \leq \rho n \), we claim that \( \deg_{H'}(u) \geq \left( \frac{5}{18} + \gamma \right) n^2 \) for every vertex \( u \in V(H') \), where \( H' := H[V \setminus V(M')] \). Indeed, from the definition of \( W' \), \( \deg_{H'}(u) > (5/18 + \tau)n^2 \) for every vertex \( u \in V(H') \). Hence,

\[
\deg_{H'}(u) \geq \deg_{H}(u) - 3n|W'| > \left( \frac{5}{18} + \tau \right) n^2 - 3n|W'|.
\]

Since \( |W'| \leq \rho n \), \( 0 < \gamma \ll \rho \ll \tau \ll 1 \) and \( n \) is sufficiently large, we have

\[
\deg_{H'}(u) > \left( \frac{5}{18} + \tau \right) n^2 - 3\rho n^2 > \left( \frac{5}{9} + \gamma \right) \left( \frac{n}{2} \right)^2.
\]

In addition, \( n \) is divisible by 3, so \( |V(H')| \) is divisible by 3. (1) implies that \( H' \) contains a perfect matching \( M'' \). Now \( M' \cup M'' \) is a perfect matching of \( H \).

Therefore, we assume that \( |W'| \geq \rho n \) in the rest of the proof. If one vertex of \( u_1, u_2, u_3 \), say, \( u_1 \), is adjacent to one vertex in \( W' \), the definition of \( W' \) implies that \( \deg(u_1) > 2n^2/3 - \varepsilon n^2 - (5/18) \tau n^2 \). Recall that \( \deg(u_i) > n^2/3 - \varepsilon n^2/2 \) for \( i = 2, 3 \). Thus

\[
\sum_{i=1}^{3} \deg(u_i) > \left( \frac{4}{3} \right) n^2 - 2\varepsilon n^2 - \left( \frac{5}{18} + \tau \right) n^2 = \left( \frac{19}{18} - 2\varepsilon - \tau \right) n^2. \tag{4}
\]

By Lemma 5, we have

\[
\sum_{i=1}^{3} \deg(u_i) \leq 18 \left( \frac{|M|}{2} \right) + 9|M| + 9|M|(n - 3|M| - 1)
\]

\[
= -18 \left( |M| - \frac{1}{4} n + \frac{1}{4} \right)^2 + \frac{9}{8} n^2 - \frac{9}{4} n + \frac{9}{8},
\]

where \( 18\left( \frac{|M|}{2} \right) \) accounts for edges between pairs of matching \( M \), \( 9|M| \) for edges with two vertices in the same matching edge from \( M \), and \( 9|M|(n - 3|M| - 1) \) for edges with one vertex in \( V(M) \), one vertex in \( U_3 \). Since \( |M| > n/3 - \eta n \), it follows that

\[
\sum_{i=1}^{3} \deg(u_i) \leq -18 \left( \frac{n}{3} - \eta n - \frac{1}{4} n + \frac{1}{4} \right)^2 + \frac{9}{8} n^2 - \frac{9}{4} n + \frac{9}{8} = (1 + 3\eta - 18\eta^2)n^2 + (9\eta - 3)n.
\]

However, \( (1 + 3\eta - 18\eta^2)n^2 + (9\eta - 3)n < \left( \frac{19}{18} - 2\varepsilon - \tau \right) n^2 \) because \( 0 < \varepsilon \ll \eta \ll \tau \ll 1 \) and \( n \) is sufficiently large. It contradicts (4).

If none of these three vertices \( u_1, u_2, u_3 \) are adjacent to the vertices in \( W' \), we have

\[
\sum_{i=1}^{3} \deg(u_i) \leq 18 \left( \frac{|M| - |M'|}{2} \right) + 9(|M| - |M'|) + 9(|M| - |M'|)(n - 3|M| - 1)
\]

\[
+ 3 \left( \frac{2|M'|}{2} \right) + 3(2|M'|)(n - 3|M| - 1)
\]

\[
= -3 \left( |M'| + \frac{3}{2} |M| \right)^2 - \frac{45}{4} |M|^2 + \frac{9}{2} n|M| - 9|M| + \frac{3}{4} n^2.
\]

Here \( 18\left( \frac{|M| - |M'|}{2} \right) \) accounts for edges between pairs of matching \( M \setminus M' \), \( 9(|M| - |M'|) \) for edges with two vertices in the same matching edge from \( M \setminus M' \), \( 9(|M| - |M'|)(n - 3|M| - 1) \) for edges with one vertex in \( V(M \setminus M') \), one vertex in \( U_3 \), \( 3\left( \frac{2|M'|}{2} \right) \) for edges with two vertices in \( V(M') \setminus W' \), and \( 3(2|M'|)(n - 3|M| - 1) \) for edges with one vertex in \( V(M') \setminus W' \), one vertex in \( V(H) \setminus V(M') \). Since \( -n/2 + 3|M|/2 < 0 \) and
\[ |M'| = |W'| \geq \rho n, \text{ then} \]
\[ \sum_{i=1}^{3} \deg(u_i) \leq -3 \left( \rho n + \frac{1}{2} n - \frac{3}{2} |M| \right)^2 - \frac{45}{4} |M|^2 + \frac{9}{2} n |M| - 9 |M| + \frac{3}{4} n^2 \]
\[ = -18 \left( |M| - \frac{1}{4} n - \frac{1}{4} \rho n + \frac{1}{4} \right)^2 + \left( \frac{9}{8} - \frac{15}{8} \rho^2 - \frac{3}{4} \rho \right) n^2 - \frac{9}{4} \rho n - \frac{9}{4} n^2 + \frac{9}{8}. \]

Recall that \( 0 < \rho \ll 1, \) so \( \frac{1}{4} n + \frac{1}{4} \rho n - \frac{1}{4} < \frac{\eta}{\rho} n. \) Furthermore, \( |M| > \frac{\eta}{\rho} n, \) hence we have
\[ \sum_{i=1}^{3} \deg(u_i) \leq -18 \left( \frac{n}{3} - \eta n - \frac{1}{4} n - \frac{1}{4} \rho n + \frac{1}{4} \right)^2 + \left( \frac{9}{8} - \frac{15}{8} \rho^2 - \frac{3}{4} \rho \right) n^2 - \frac{9}{4} \rho n - \frac{9}{4} n^2 + \frac{9}{8}, \]
which contradicts the condition \( \sum_{i=1}^{3} \deg(u_i) \geq 3 \left( \frac{n^2}{3} - \varepsilon n^2 / 2 \right) \) because \( 0 < \varepsilon \ll \eta \ll \rho \ll 1 \) and \( n \) is sufficiently large. This completes the proof of Theorem \[2 \]

5. Concluding remarks

In this paper we consider the minimum degree sum of two adjacent vertices that guarantees a perfect matching in 3-graphs. Given \( 3 \leq k < n \) and \( 2 \leq s \leq n/k, \) we can generalize this problem to \( k \)-graphs not containing a matching of size \( s'. \) For \( 1 \leq \ell \leq k, \) let \( H_{n,k,s}' \) denote the \( k \)-graph whose vertex set is partitioned into two sets \( S \) and \( T \) of size \( n - s \ell + 1 \) and \( s \ell - 1, \) respectively, and whose edge set consists of all the \( k \)-sets with at least \( \ell \) vertices in \( T. \) Apparently \( H_{n,k,s}' \) contains no matching of size \( s. \) A well-known conjecture of Erdős \[3\] says that \( H_{n,k,s}^1 \) or \( H_{n,k,s}^k \) is the densest \( k \)-graph on \( n \) vertices not containing a matching of size \( s. \) It is reasonable to speculate that the largest \( \sigma_2'(H) \) among all \( k \)-graphs \( H \) on \( n \) vertices not containing a matching of size \( s \) is also attained by \( H_{n,k,s}' \). Note that \( H_{n,k,s}' \) is a complete \( k \)-graph of order \( sk - 1 \) together with \( n - sk + 1 \) isolated vertices and thus \( \sigma_2'(H_{n,k,s}') = 2 \left( \frac{s k - 2}{k - 1} \right). \) When \( 1 \leq \ell \leq k - 2, \) any two vertices of \( H_{n,k,s}' \) are adjacent and thus \( \sigma_2'(H_{n,k,s}') = 2 \delta_1(H_{n,k,s}'). \) When \( \ell = k - 1, \) it is easy to see that \( \sigma_2'(H_{n,k,s}'_{k-1}) = 2 \left( (k-1)-2 \right) + (n - s(k-1) + 2) \left( (s(k-1)-2) \right). \]

Assume \( s = n/k. \) Since \( H_{n,k,n/k}^k \) contains isolated vertices and \( \delta_1(H_{n,k,n/k}^k) \leq \delta_1(H_{n,k,n/k}') \) for \( 1 \leq \ell \leq k - 2, \) we only need to compare \( \sigma_2'(H_{n,k,n/k}) \) and \( \sigma_2'(H_{n,k,n/k}') \). For sufficiently large \( n, \) it is easy to see that \( \sigma_2'(H_{n,k,n/k}^1) < \sigma_2'(H_{n,k,n/k}') \) when \( k \leq 6 \) and \( \sigma_2'(H_{n,k,n/k}^k) > \sigma_2'(H_{n,k,n/k}') \) when \( k \geq 7. \)

Problem 9. Does the following hold for any sufficiently large \( n \) that is divisible by \( k? \) Let \( H \) be a \( k \)-graph of order \( n \) without isolated vertex. If \( k \leq 6 \) and \( \sigma_2'(H) > \sigma_2'(H_{n,k,n/k}') \) or \( k \geq 7 \) and \( \sigma_2'(H) > \sigma_2'(H_{n,k,n/k}^1), \) then \( H \) contains a perfect matching.

Now assume \( k = 3 \) and \( 2 \leq s \leq n/3. \) Note that
\[ \sigma_2'(H_{n,3,s})^3 = 2 \left( \frac{3s - 2}{2} \right), \quad \sigma_2'(H_{n,3,s})^1 = 2 \left( \frac{n - 1}{2} - \frac{n - s}{2} \right), \quad \text{and} \]
\[ \sigma_2'(H_{n,3,s})^2 = \left( \frac{2s - 2}{2} + (n - 2s + 1) \left( \frac{2s - 2}{2} + 1 \right) + \left( \frac{2s - 1}{2} + 2 \right) (2s - 2)(n - 1). \]

It is easy to see that \( \sigma_2'(H_{n,3,s})^3 > \sigma_2'(H_{n,3,s})^1. \) Zhang and Lu \[23\] made the following conjecture.

Conjecture 10. \[23\] There exists \( n_0 \in \mathbb{N} \) such that the following holds. Suppose that \( H \) is a 3-graph of order \( n \geq n_0 \) without isolated vertex. If \( \sigma_2'(H) > 2 \left( \frac{n - 1}{2} - \frac{n - s}{2} \right) \) and \( n \geq 3s, \) then \( H \) contains no matching of size \( s \) if and only if \( H \) is a subgraph of \( H_{n,3,s}^3. \)

Zhang and Lu \[23\] showed that the conjecture holds when \( n \geq 9s^2. \) Later the same authors \[24\] proved the conjecture for \( n \geq 13s. \) If Conjecture 10 is true, then it implies the following theorem of Kühn, Osthus and Treglown \[10\].
Theorem 11. [10] There exists \( n_0 \in \mathbb{N} \) such that if \( H \) is a 3-graph of order \( n \geq n_0 \) with \( \delta_1(H) \geq \binom{n-1}{2} - \binom{n-3}{2} + 1 \) and \( n \geq 3s \), then \( H \) contains a matching of size \( s \).

Our Theorem 1 suggests a weaker conjecture than Conjecture 10.

Conjecture 12. There exists \( n_1 \in \mathbb{N} \) such that the following holds. Suppose that \( H \) is a 3-graph of order \( n \geq n_1 \) without isolated vertex. If \( \sigma_2^2(H) > \sigma_2^2(H_{n,3,s}^2) \) and \( n \geq 3s \), then \( H \) contains a matching of size \( s \).

On the other hand, we may allow a 3-graph to contain isolated vertices. Note that \( \sigma_2^2(H) > \sigma_2^2(H_{n,3,s}^2) \) if and only if \( s \leq (2n+4)/9 \). We make the following conjecture.

Conjecture 13. There exists \( n_2 \in \mathbb{N} \) such that the following holds. Suppose that \( H \) is a 3-graph of order \( n \geq n_2 \) and \( 2 \leq s \leq n/3 \). If \( \sigma_2^2(H) > \sigma_2^2(H_{n,3,s}^2) \) and \( s \leq (2n+4)/9 \) or \( \sigma_2^2(H) > \sigma_2^2(H_{n,3,s}^2) \) and \( s > (2n+4)/9 \), then \( H \) contains a matching of size \( s \).

In fact, we can derive Conjecture 13 from Conjecture 12 as follows. Let \( n_2 = \max\{\binom{n}{2}, \frac{3}{2}n_1\} \) and \( H \) be a 3-graph of order \( n \geq n_2 \) satisfying the assumption of Conjecture 13. If \( H \) contains no isolated vertex, then \( H \) contains a matching of size \( s \) by Conjecture 12. Otherwise, let \( W \) be the set of isolated vertices in \( H \). Let \( H' = H[V(H) \setminus W] \) and \( n' = n - |W| \). Then \( H' \) is a 3-graph without isolated vertex and \( \sigma_2^2(H') = \sigma_2^2(H) \).

When \( 2 \leq s \leq (2n+4)/9 \), we have \( \sigma_2^2(H') > \sigma_2^2(H_{n,3,s}^2) > \sigma_2^2(H_{n',3,s}^2) \). In addition, since \( n \geq \binom{n}{2} \) and
\[
2 \left( \frac{n' - 1}{2} \right) \geq \sigma_2^2(H') > 2(s-2)(n-1) \geq 2(n-1),
\]
we have \( n' \geq n_1 \). When \( s > (2n+4)/9 \), we have \( \sigma_2^2(H') > \sigma_2^2(H_{3,n,3,s}^2) > \sigma_2^2(H_{n,3,s}^2) > \sigma_2^2(H_{n',3,s}^2) \). In addition, since \( n \geq 3n_1/2 \) and
\[
2 \left( \frac{n' - 1}{2} \right) \geq \sigma_2^2(H') > 2\left( \frac{3s-2}{2} \right) > 2\left( \frac{2(n-1)/3}{2} \right),
\]
we have \( n' \geq n_1 \). In both cases, Conjecture 12 implies that \( H' \) contains a matching of size \( s \).

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