Hamiltonian Cycles With All Small Even Chords

Guantao Chen, Katsuhiro Ota, Akira Saito, and Yi Zhao

Abstract. Let G be a graph of order $n \geq 3$. A subgraph H of G is called a *square hamiltonian cycle* if it consists of a hamiltonian cycle $v_1v_2 \ldots v_nv_1$ and chords v_iv_{i+2} for all $i = 1, 2, \ldots, n$ (where $v_{n+j} = v_j$ for $1 \leq j \leq n$). Clearly, a square hamiltonian cycle contains all possible 2-regular graphs of order n. Fan and Kiestead showed that, for every positive real number $\varepsilon > 0$, if minimum degree $\delta(G) \geq (\frac{2}{3} + \varepsilon)n$ then G contains a square hamiltonian cycle. Komlés, Sárközy, and Szemerédi showed that the ε in the condition can be removed for graphs with large number of vertices. A subgraph H of G is called an *even square hamiltonian cycle* (ESHC) if it consists of a hamiltonian cycle $C = v_1v_2 \ldots v_nv_1$ of G and chords v_iv_{i+3} for all $1 \leq i \leq n$. Clearly, an ESHC contains all possible 2-regular graphs of order n with *even* components. We prove that there is a positive integer N such that, for every graph G of even order $n \geq N$, if minimum degree $\delta(G) \geq \frac{1}{2}(n + 614)$ then G contains an ESHC. Using the well-known fact that there are infinite many graphs with large minimum degree without C_4s, we construct infinitely many examples demonstrating that the condition of n to be even cannot be dropped. The coefficient $\frac{1}{2}$ is the best possible but the constant 614 may be not. There are infinity many graphs showing that the minimum degree condition cannot be lower to $\frac{1}{2}(n + 4)$. We believe that the number 614 may be reduced to 4. We also generalized the notion of *ESHC* to *kth power hamiltonian cycle* and obtained similar results.

1. Introduction

In this paper, we will only consider simple graphs – finite graphs without loops or multiple edges. We will follow Diestel [**7**] for notation and terminology not defined here. Let $G = (V, E)$ be a graph with vertex set V and edge set E. For a vertex $v \in V$, let $\Gamma(v)$ and $\deg(v)$ denote the

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neighborhood and degree of v, respectively. More specifically, for a subgraph $H \subseteq G$, let $\Gamma(v, H)$ and $\deg(v, H)$ denote the set of neighbors and degree of v in H, respectively. For any $U \subseteq V$, let $\Gamma(U, H) := \cap_{u \in U} \Gamma(u, H)$ and $\deg(U, H) := |\Gamma(U, H)|$. A graph G is called *hamiltonian* if it contains a spanning cycle. The Hamiltonian Problem, determining whether a graph has a hamiltonian cycle, has long been one of few fundamental problems in graph theory. We refer Gould [**13**] for the most recent survey in this area. Let G be a graph of order $n \geq 3$. Dirac [**8**] proved that if minimum degree $\delta(G) \geq n/2$ then G is hamiltonian. Ore [**25**] extended Dirac's result by replacing the minimum degree condition with that $\deg(u) + \deg(v) \geq n$ for every pair of nonadjacent vertices u and v. Many results have been obtained on generalizing these two classic results. Clearly, the *denser* is the graph the stronger hamiltonicity properties are revealed.

A hamiltonian cycle of a graph G is a *connected* 2-factor of the graph. Aigner and Brandt [**2**] proved that if minimum degree $\delta(H) \geq \frac{2n-1}{3}$ then G contains graphs of maximum degree 2 as subgraphs, which was conjectured by Sauer and Spencer [**26**], a special case of the packing conjecture of Bollobás and Eldridge. The following conjecture specifying the lengths of each components is still open.

Conjecture 1.1 (El-Zahar [**9**]). *Let G be a graph of order n. For any partition $n = n_1 + n_2 + \cdots + n_k$ with $n_k \geq n_2 \geq \cdots \geq n_1 \geq 3$, if $\delta(G) \geq \sum_{i=1}^{k} \lceil \frac{n_i}{2} \rceil$ then G has a 2-factor consisting of cycles $C_1, C_2, \ldots, C_k$ of lengths $n_1, n_2, \ldots, n_k$, respectively.*

The conjecture is equivalent to say that, if $\delta(G) \geq (n+k)/2$, then G contains all 2-regular graphs with at most k odd components. By using the Szemerédi regularity lemma, Abbasi [**1**] announced the proof of Conjecture 1.1 for graphs with large number of vertices. Another approach to the Sauer and Spencer Conjecture is to show that the considered graphs contain a specific graph which contains all 2-regular graphs. A *square hamiltoinian cycle* of a graph G is a hamiltonian cycle $v_1 v_2 \ldots v_n v_1$ plus edges $v_i v_{i+2}$ for all $0 \leq i \leq n-1$, where $v_{n+i} = v_i$ for $i = 1, 2$. Clearly, a square hamiltonian cycle of G contains all possible of n-vertex graphs of maximum degree at most 2. Posá [**10**] conjectured that, *for a graph G of order $n \geq 3$, if minimum degree $\delta(G) \geq \frac{2}{3}n$ then G contains a square hamiltonian cycle.* Fan and Kiestead [**11**] set the conjecture approximately. Komlés, Sárközy, and Szemerédi [**18**] established the Posás Conjecture for graphs with large number of vertices. The notion of square hamiltonian cycles has been generalized to *k-th power hamiltonian cycle*, a hamiltonian cycle $C := v_1 v_2 \ldots v_n v_1$ plus all possible chords $v_i v_j$ whenever $|j - i| \leq k$. Posá-Seymour conjectured that every n-vertex graph G with minimum degree $\delta(G) \geq (k-1)n/k$ contains a k-th power hamiltonian cycle. Komlés, Sárközy, and Szemeréd in [**22, 21**] using the Regularity Lemma and the Blow-up Lemma established the conjecture.

Let b be positive integer. A graph H is said to have *bandwidth* at most b, if there exists a labeling of the vertices by v_1, v_2, …, v_n, such that $|j-i| \leq b$ if $v_iv_j \in E(H)$. Analogously to the Erdős-Stone theorem, Bollobás and Komlós [**16**](Conjecture 16) conjectured that, if G is an n-vertex graph with minimum degree $\delta(G) \geq (\frac{r-1}{r}+\varepsilon)n$ then G contains every n-vertex r-chromatic graph with bounded degree and bandwidth $b = o(n)$. The conjecture was recently settled by Böttcher, Schacht and Taraz [**4**]. There are infinity many graphs showing that the coefficient $(k-1)/k$ is best possible. We are interested in whether the constant ε can be dropped.

In this paper, we consider 2-regular graphs with all components having even number vertices. According to the El-Zahar Conjecture, every n-vertex graph G with minimum degree $\delta(G) \geq n/2$ should contain all such subgraphs. We call a subgraph H of a graph G an *Even Square Hamiltonian Cycle* (ESHC) if it consists of a hamiltonian cycle $C := v_1v_2\dots v_nv_1$ of G and all chords v_iv_{i+3} for all $1 \leq i \leq n$, where $v_{i+n} = v_i$ for all $1 \leq i \leq n$, where C is called *the support cycle* of the ESHC H. Clearly, an n-vertex ESHC contains all graphs of order n with maximum degree ≤ 2 and chromatic number 2. Moreover, an ESHC itself has maximum degree $\Delta = 4$ and chromatic number $\chi = 2$. The main result of this paper is stated below.

Theorem 1.2. *There is a universal constant s and a positive integer $N := N(s)$ such that. if G is a graph of order $n \geq N$ such that n is even and and minimum degree $\delta(G) \geq \frac{1}{2}(n+s)$, then G contains an ESHC. Moreover, we can take $s = 614$.*

Let G be a graph. A pair (A, B) of induced subgraphs of G is called a *separator* of G if $V(A) \cup V(B) = V(G)$, $A - B \neq \emptyset \neq B - A$ and $E(A - B, B - A) = \emptyset$. Clearly, $A \cap B$ is a vertex cut of G if (A, B) is a separator of G. The following result shows that the universal constant s in the above theorem must be at least 4, which we believe is best possible.

Proposition 1.3. *Let G be a graph containing an EHSC. If (A, B) is a separator of G such that $|A - B| \geq 3$ and $|B - A| \geq 3$, then $|A \cap B| \geq 6$. Consequently, if G is the union of two copies of $K_{\frac{n+4}{2}}$ sharing at most 4 vertices and $n \geq 12$, then G does not contain an ESHC while minimum degree $\delta(G) = \frac{n+2}{2}$.*

Proof. Let G be a graph, H be an ESHC of G, and C be the support cycle of H. Assign C an orientation and let $P_1 := P_1[x_1, z_1]$ and $P_2 := P_2[x_2, z_2]$ be two disjoint segments of C such that the internal-vertices of

$P_i(x_i, z_i)$ are in $A \cap B$ and two end vertices x_i and z_i are in $A - B$ and $B - A$, respectively, for each $i = 1, 2$. Since C is the support cycle of the ESHC H and $x_i z_i \notin E$, we have $|P_i(x_i, z_i)| \neq 2$ for each $i = 1, 2$. If $|P_i(x_i, z_i)| \geq 3$ for both $i = 1$ and 2, then $|A \cap B| \geq |P_1(x_1, z_1)| + |P_2(x_2, z_2)| \geq 6$. We assume, without loss of generality, $P_1(x_1, z_1) = y_1$ and $P_1[x_1, z_1]$ is along the orientation of C. Let x_1^- be the predecessor of x_1 along C and z_1^+ be the successor of z_1 along C. Since $x_1^- z_1 \in E(G)$, we have $x_1^- \in A \cap B$. Similarly, $z_1^+ \in A \cap B$. Since P_1 and P_2 are vertex disjoint, $V(P_2(x_2, z_2)) \cap \{x_1^-, y_1, z_1^+\} = \emptyset$. If $|P_2(x_2, z_2)| \geq 3$, then $|A \cap B| \geq 6$. So we may assume that $P_2(x_2, z_2) = y_2$ and $\{x_2^-, y_2, z_2^+\} \subseteq A \cap B$, where x_2^- is the predecessor of P_2 along C and z_2^+ is the successor of P_2 along C. Since $|A - B| \geq 3$ and $|B - A| \geq 3$, there is a path $P_3 := P_3[x_3, z_3]$ disjoint from $P_1 \cup P_2$ such that $V(P_3(x_3, z_3)) \subseteq A \cap B$ and x_3 and z_3 are in $A - B$ and $B - A$, respectively. Similar, we have $P_3(x_3, z_3) = y_3$ and $\{x_3^-, y_3, z_3^+\} \subseteq A \cap B$, where x_3^- is the predecessor of P_3 along C and z_3^+ is the successor of P_3 along C. Since P_1, P_2 and P_3 are segments along C, we have $|\{x_i^-, y_i, z_i^+\} \cap \{x_j^-, y_j, z_j^+\}| \geq 1$ for any $1 \leq i \neq j \leq 3$. Thus, $|A \cap B| \geq |\cup_{1 \leq i \leq 3} \{x_i^-, y_i, z_i^+\}| \geq 6$. $\square$

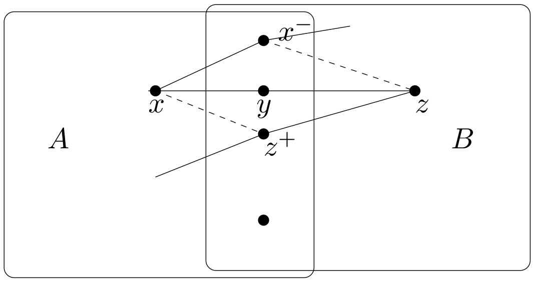


FIGURE 1. A segment connecting A and B

The following result shows that the condition *that n is even* necessary.

Proposition 1.4. *For any positive integer s, there exist infinitely many graphs G of* **odd** *order n and minimum degree $\delta(G) \geq (n + s)/2$ but does not contain an ESHC.*

PROOF. For any positive integer s, by a well-known result of Erdős, there are infinitely many graphs H such that minimum degree $\delta(H) > s$ and girth $g(H) \geq 5$. Let H be one of such graph such that $h := |V(H)| > s$. Let $G := H + \overline{K}_{h-s}$, i.e. a graph obtained from H by adding $h - s$ vertices such that each new vertex is adjacent to every vertex of H. Let $X := V(G) - V(H)$ and $n := |V(G)| = 2h - s$. Clearly, $\delta(G) \geq h = \frac{n+s}{2}$. To show that G does not have an ESHC, let C be an arbitrary hamiltonian cycle of G (if there exists one). Clearly, $C - X$ is a union of vertex-disjoint paths in H. Since X is an independent vertex set and $|C| = n$ is odd, one of the such segments $P := [x, y]$ contains even number of vertices. If $|V(P)| \geq 4$ then $xx^{+++} \notin E(G)$ since $g(H) \geq 5$. Since $|P|$ is even, then $|V(P)| = 2$, which in turn shows that $x^-, y^+ \in X$. Since X is independent, $x^-y^+ \notin E(G)$. In any case, we have shown that G does not have an ESHC based on C. □

More generally, an *Even kth Hamiltonian Cycle* (EkHC) of a graph G is a hamiltonian cycle $v_1v_2 \dots v_nv_1$ plus edges v_iv_{i+2j-1} for all integers $1 \leq j \leq k$. Clearly, an E1HC is a hamiltonian cycle and an E2HC is an ESHC. Using the same argument as the proof of Theorem 1.2, with much more tedious details, we obtain the following result.

Theorem 1.5. *For any positive integer k, there exist a universal constant $s := s(k)$ and a such that a positive integer $N := N(k)$ such that if G is a graph of even order $n \geq N$ and $\delta(G) \geq \frac{n+s}{2}$ then G contains an EkHC.*

An *Even Square Path* (ESP) of length t is a path $v_0v_1 \dots v_t$ plus all edges v_iv_{i+3} for each $1 \leq i \leq t-3$. In the proof, we will concatenate special ESPs to form an ESHC.

2. The Regularity Lemma and The Blow-up Lemma

The Regularity Lemma [**27**] and the Blow-up Lemma [**19, 20**] are main tools in the proof of Theorem 1.2. Let ε and δ be two positive real numbers. The notion of ε-regularity and (ε, δ)-super-regularity first describes the randomized properties of a graph. Let $G = (V, E)$ be a graph. For any two disjoint vertex-sets A and B, let $e(A, B)$ denote the number of edges with one endvertex in A and the other in B; the *density* of A and B is the *ratio* $d(A, B) := e(A, B)/(|A| \cdot |B|)$; the pair (A, B) is *ε-regular* (otherwise *ε-irregular*) if for every $X \subseteq A$ and $Y \subseteq B$, satisfying $|X| > \varepsilon|A|$, $|Y| > \varepsilon|B|$, we have $|d(X, Y) - d(A, B)| < \varepsilon$. Moreover, The pair (A, B) is (ε, δ)-*super-regular* if (A, B) is ε-regular and $\deg_B(a) > \delta|B|$ for all $a \in A$ and $\deg_A(b) > \delta|A|$ for all $b \in B$.

Lemma 2.1 (Regularity Lemma – Degree Form)**.** *For every $\varepsilon > 0$ there is an $M = M(\varepsilon)$ such that, for any graph $G = (V, E)$ and any real number $d \in [0, 1]$, there is a partition of the vertex set V into $\ell + 1$ clusters $V_0, V_1, \dots, V_\ell$, and there is a subgraph G' of G with the following properties:*

- $\ell \leq M$,
- $|V_0| \leq \varepsilon|V|$,
- $|V_1| = |V_2| = \cdots = |V_\ell| \leq \lceil \varepsilon|V| \rceil$,
- $\deg_{G'}(v) > \deg_G(v) - (d + \varepsilon)|V|$ *for all* $v \in V$,

- *$G'[V_i] = \emptyset$ (i.e. V_i is an independent set in G'), for all i,*
- *each pair (V_i, V_j), $1 \le i < j \le \ell$, is ε-regular with density $d(V_i, V_j)$ either 0 or $\ge d$ in G'.*

The Blow-up Lemma of Komlós, Sárközy and Szemerédi [**19**] allows us to regard the super-regular pairs in G as complete bipartite graphs when embedding to G a graph with bounded degree. We will need a special version of this lemma which restricts the mappings of constant many vertices.

Lemma 2.2 (Blow-up Lemma)**.** *Let R be graph of order r, $0 < \delta, \alpha \le 1$ be two real numbers, and Δ and τ be two positive integers. Then there exists an $\varepsilon > 0$ such that the following holds:*

Let N be an arbitrary positive integer, and replace the vertices of R with pairwise disjoint N-sets $V_1, V_2, \ldots, V_r$ (blowing up). Let $R(N)$ and G be two graphs on the same vertex-set $V = \cup V_i$ such that the edge set of $R(N)$ is obtained by replacing all edges of R with copies of the complete bipartite graph $K_{N,N}$ and the graph G is constructed by replacing the edges of R with some (ε, δ)-super-regular pairs. If a graph H with maximum degree $\Delta(H) \le \Delta$ can be embedded into $R(N)$, then it can be embedded into G. Moveover, for each i fix τ special vertices x to be embedded into V_i, and for x fix a set $S_x \subset V_i$ of size at least αN. The embedding of H holds even if we restrict the image of x to be S_x for all x.

3. Technic Theorems

The following results are needed in the proof.

Theorem 3.1 (Nash-Williams[**24**])**.** *Let G be a 2-connected graph of order n. If minimum degree $\delta(G) \ge \max\{(n+2)/3, \alpha(G)\}$, then G contains a hamiltonian cycle.*

Theorem 3.2 (Moon-Morser[**23**])**.** *Let $G = (X, Y; E)$ be a bipartite graph with $|X| = |Y| = p$. If $\delta(G) > p/2$, then G contains a hamiltonian cycle.*

In fact, the following stronger version of the Moon-Morser Theorem will be needed. We omit the proof here since its proof is standard.

Theorem 3.3. *Let k be a positive integer and $G = (X, Y; E)$ be a bipartite graph with $|X| = |Y| = p$. If $\delta(G) > \frac{p+k}{2}$, then, for any k edges whose union forms a family of disjoint paths, there is a hamiltonian cycle containing these k edges.*

Lemma 3.4. *For any real number $\alpha \in (0,1)$ and a set X, let $\mathcal{F} = \{X_1, X_2, \ldots, X_m\}$ be a family of subsets of X such that $|X_i| \geq \alpha|X|$ and $m \geq 2/\alpha$. Then there exist $1 \leq i \neq j \leq m$ such that $|X_i \cap X_j| \geq \frac{\alpha^2}{2}|X|$.*

Proof. Lemma 3.4 follows from the inequality $|X| \geq |\cup_{i=1}^m X_i| \geq \sum_{i=1}^m |X_i| - \sum_{1\leq i<j\leq m} |X_i \cap X_j|$. $\square$

Lemma 3.5. *For any real number $\alpha \in (0,1)$, there exists a real number $\beta > 0$ and a positive number M such that the following hold.*

Let (X,Y) be a pair of sets of n elements, $m \geq M$ be a positive integer, $\mathcal{X} = \{X_1, X_2, \ldots, X_m\}$ be a family of subsets of X such that $|X_i| \geq \alpha|X|$ for each $1 \leq i \leq m$, and $\mathcal{Y} = \{Y_1, Y_2, \ldots, Y_m\}$ be a family of subsets of Y such that $|Y_i| \geq \alpha|Y|$ for each $1 \leq i \leq m$. Then there exist two distinct pairs (X_i, Y_i) and (X_j, Y_j) such that $|X_i \cap X_j| \geq \beta|X|$ and $|Y_i \cap Y_j| \geq \beta|Y|$.

Proof. Let M be the Ramsey number $r(\lceil \frac{2}{\alpha} \rceil, \lceil \frac{2}{\alpha} \rceil)$ and let $\beta = \alpha^2/2$. From Lemma 3.4, we know there do not exist M sets of size $\alpha|X|$ such that the intersection any two is less than $\beta|X|$. Applying the Ramsey theorem, we obtain a subfamily $\mathcal{X}^* \subseteq \mathcal{X}$ of size M such that $|X_i \cap X_j| \geq \beta|X|$ for each pair X_i, $X_j \in \mathcal{X}^*$. Applying Lemma 3.4 for the corresponding Y_i, we have Lemma 3.5. $\square$

Lemma 3.6. *Let G be a graph with independence number $\alpha(G) \leq \eta$. Then there exist η vertex disjoint paths P_1, P_2, $\ldots$, P_η such that $V(P_1) \cup V(P_2) \cup \cdots \cup V(P_\eta) = V(G)$.*

Proof. let P_1, P_2, $\ldots$, P_η be η vertex disjoint paths of G such that $\sum_{i=1}^{\eta} |P_i|$ is maximum. We claim $\cup_{i=1}^{\eta} V(P_i) = V(G)$. Otherwise, let $x \in V(G) - \cup_{i=1}^{m} V(P_i)$ and x_i be one of the endvertices of P_i for each $1 \leq i \leq \eta$. It is readily seen that $\{x, x_1, x_2, \ldots, x_\eta\}$ is an independent set of G, which contradicts that $\eta = \alpha(G)$. $\square$

Lemma 3.7. *In a graph G of order n with the maximum degree Δ and the minimum degree δ, the number of disjoint 4-stars is at least $\frac{(\delta-3)n}{5(\Delta+\delta-3)}$.*

Proof. Suppose G has a largest family of disjoint 4-stars on M of size m. Then $(\delta-3)(n-5m) \leq e(M, V(G)-M) \leq 5m\Delta$ and the claim follows. $\square$

4. The Proof of Theorem 1.2

The proof of Theorem 1.2 is divided into two main cases: Non-extremal Case and Extremal Case, which are presented in Sections.

4.1. Non-extremal Case.

In this section we prove Theorem 1.2 under *non-extremal* case. There are two extremal cases defined below.

Extremal Case 1: There exists a balanced partition of V into V_1 and V_2 such that the density $d(V_1, V_2) \geq 1 - \alpha$ for some small α. We will explain a desired value of α in the proof of this case.

Extremal Case 2: There exists a balanced partition of V into V_1 and V_2 such that the density $d(V_1, V_2) \leq \alpha$ for some small α. We will explain a desired value of α in the proof of this case.

Theorem 4.1. *For every $\alpha > 0$, there exist $\beta > 0$ and a positive integer n_0 such that, for every graph G of order $n \geq n_0$ and $\delta(G) \geq (\frac{1}{2} - \beta)n$, G either contains an ESHC or is in one of the extremal cases with parameter α.*

Proof. We fix the following sequence of parameters

$$\varepsilon \ll d \ll \beta \ll \alpha$$

and specify the actual dependence of them as proof proceed. Here is a highlight on how are they related. 1. We pick α be a small universal positive real number to handle two extremal cases. 2. We pick β and δ to force two extremal cases while ε is choose to handle the non-extremal case. 3. ε is chosen such that the Blow-up lemma (Lemma 2.2) can be applied for an $(\varepsilon, d/2)$-super-regular pair (X, Y).

Choose n to be a very large integer. Let G be a graph of order n such that $\delta(G) \geq (\frac{1}{2} - \beta)n$ and G is **not** in either of the extremal cases (with α). Applying the Regularity Lemma (Lemma 2.1) to G with parameters ε and d, we obtain a positive integer $M := M(\varepsilon)$ and a partition of the vertex set V into $\ell + 1$ clusters $V_0, V_1, \dots, V_\ell$, and there is a subgraph G' of G with the following properties:

- $\ell \leq M$,
- $|V_0| \leq \varepsilon |V|$,
- $|V_1| = |V_2| = \cdots = |V_\ell| \leq \lceil \varepsilon |V| \rceil$,

- $\deg_{G'}(v) > \deg_G(v) - (d+\varepsilon)n \geq (\frac{1}{2} - \beta - \varepsilon - d)n \geq (\frac{1}{2} - 2\beta)n$ for all $v \in V$, consequently, $e(G') \geq e(G) - \frac{(d+\varepsilon)}{2}n^2 \geq e(G) - dn^2$, (since $\varepsilon < d$)
- $G'[V_i] = \emptyset$ (i.e. V_i is an independent set in G'), for all i,
- each pair (V_i, V_j), $1 \leq i < j \leq \ell$, is ε-regular with density $d(V_i, V_j)$ either 0 or $\geq d$ in G'.

We may assume that $\ell = 2k$ is even; otherwise we eliminate the last cluster V_ℓ by removing all the vertices in V_ℓ to V_0. As a result, $|V_0| \leq 2\varepsilon n$.

For each pair i and j with $1 \leq i \neq j \leq \ell$, we write $V_i \sim V_j$ if $d(V_i, V_j) \geq d$. As in many applications of the Regularity Lemma, it is convenient to consider the *reduced graph* G_r. Each vertex $i \in V(G_r)$ corresponds to V_i; two vertices i and j are adjacent if and only if $V_i \sim V_j$. Since $\delta(G') > (\frac{1}{2} - 2\varepsilon - d)n > (\frac{1}{2} - 3\varepsilon)n$, we have that that $\delta(G_r) \geq (\frac{1}{2} - 3\beta)\ell$.

Claim 4.2. *The following two statements hold.*

(1) *If G_r contains an independent set U_1 such that $|U_1| \geq (\frac{1}{2} - 10\beta)\ell$, then G is in the Extremal Case 1 with parameter α.*

(2) *If G_r contains two disjoint subsets U_1, U_2 such that both $|U_i| \geq (\frac{1}{2} - 6\beta)\ell$ (for each $i = 1, 2$), and no edge exists between U_1 and U_2, then G is in the Extremal Case 2 with parameter α.*

Either case leads a contradiction to our assumption.

Proof. **(1).** To avoid cumbersome notation and calculation, we assume $(\frac{1}{2} - 10\beta)\ell$ is an integer and will do the similar assumptions through the proof. Let $A = \bigcup_{i \in U_1} V_i$ and $B = V(G) - A$. Since $(1-\varepsilon)n < N\ell \leq n$, then

$$(\frac{1}{2} - 11\beta)n \leq |A| = |U_1|N = (\frac{1}{2} - 10\varepsilon)N\ell \leq (\frac{1}{2} - 2\beta)n.$$

For each $x \in A$, since $\deg_G(x, A) \leq \deg_{G'}(x, A) + (d + \varepsilon)n < \beta n$ (because $d+\varepsilon < d$), then $\deg_G(x, B) > (\frac{1}{2} - \beta)n - \beta n > (\frac{1}{2} - 2\beta)n$. Hence $e_G(A, B) > (\frac{1}{2} - 11\beta)n(\frac{1}{2} - 2\beta)n > (\frac{1}{4} - \frac{13}{2}\beta)n^2$. Now move at most $11\beta n$ vertices from B to A such that A and B are of size $n/2$. We still have $e_G(A, B) > (\frac{1}{4} - \frac{13}{2}\beta)n^2 - 11\beta nn/2 > (1 - 48\beta)|A||B|$. By imposing the condition $48\beta < \alpha$, we see that G is in the Extremal Case 1 with parameter α.

(2). By taking subsets if necessary, we may assume that $|U_1| = |U_2| = (\frac{1}{2} - 6\beta)\ell$. Let $A = \bigcup_{i \in U_1} V_i$ and $B = \bigcup_{i \in U_2} V_i$. Since there is no edge between U_1 and U_2, we have $e_{G'}(A, B) = 0$. When we apply the Regularity Lemma, if $d(V_i, V_j) \geq d$ we can keep the edge set $E(V_i, V_j)$ in G'. So

$d(V_i, V_j) < d$ if $ij \notin E(G_r)$. Consequently $e_G(A, B) \leq e_{G'}(A, B) + dn^2 = dn^2$. Note that $|A| = |U_1|N = (\frac{1}{2} - 6\beta)\ell N > (\frac{1}{2} - 7\beta)n$. Similarly, $|B| > (\frac{1}{2} - 7\beta)n$. By adding at most $7\beta n$ vertices to each of A and B, we obtain two subsets of size $n/2$ and still name them as A and B, respectively. Then, $e(A, B) \leq dn^2 + 2 \cdot (7\beta n)(n/2) = (d + 7\beta)n^2$, which in turn shows the density $d(A, B) \leq 2(d + 7\beta)$. By choosing $\alpha > 2(d + 7\beta)$, we obtain that G is in the Extremal Case 2 with parameter α. □

Claim 4.3. *G_r is hamiltonian.*

Proof. We first show that G_r is $\beta\ell$-connected. (Again, we assume that $\beta\ell$ is an integer to avoid cumbersome notation.) Suppose, to the contrary, let S be a cut of G_r such that $|S| < \beta\ell$ and let U_1 and U_2 be two components of $G_r - S$. Since $\delta(G_r) \geq (\frac{1}{2} - 2\beta)\ell$, $|U_i| \geq (\frac{1}{2} - 3\beta)\ell$ for each $i = 1, 2$. By Claim 4.2 **(2)**, G is in the Extremal Case 2, a contradiction. Since $n = \sum_{i=0}^{\ell} |V_i| \leq (\ell + 2)\varepsilon n$, we have $\ell \geq 1/\varepsilon - 2 \geq 3/\beta$. So $\beta\ell \geq 3$, which gives that G_r is 3-connected.

By Claim 4.2 **(1)**, the independence number of G_r is less than $(\frac{1}{2} - 6\varepsilon)\ell$, which is less than the minimum degree of $\delta(G_r) \geq (\frac{1}{2} - 2\beta)\ell$. By Theorem 3.1, G_r is hamiltonian. □

Following the order of a hamiltonian cycle of G_r, we denote all the clusters of G except for V_0 by $X_1, Y_1, \ldots, X_k, Y_k$ (recall that $\ell = 2k$ is even). We call X_i the partner of Y_i and vice versa. For each $i = 1, \ldots k$, we initiate an ESP (even square path) connecting X_i and Y_{i-1} ($Y_0 = Y_k$) as follows, where $Y_0 = Y_k$.

Given an ε-regular pair (X, Y) of clusters and $Y' \subseteq Y$, we call a vertex $x \in X$ *typical* to Y' if $\deg(x, Y') \geq (d - \varepsilon)|Y'|$. By the regularity of (X, Y), at least $(1 - \varepsilon)N$ vertices of X are typical to Y' when $|Y'| \geq \varepsilon N$.

For each $i = 1, 2, \ldots, k$, let $a_i \in X_i$ be a vertex typical to both Y_{i-1} and Y_i and let $b_i \in X_i$ be a vertex typical to both $\Gamma(a_i, Y_{i-1})$ and $\Gamma(a_i, Y_i)$. Since both pairs (Y_{i-1}, X_i) and (X_i, Y_i) are ε-regularity and density at least $d > \varepsilon$, where $Y_0 = Y_k$. Such two vertices a_i and b_i exist. Then, for each $i = 1, 2, \ldots, k$, let $c_i, d_i \in \Gamma(\{a_{i+1}, b_{i+1}\}, Y_i)$ be two vertices such that c_i is typical to X_i and d_i is typical to X_{i+1} and $\Gamma(c_i, X_i)$. (where $\Gamma(\{a_i, b_i\}, Y_i) = \Gamma(a_i, Y_i) \cap \Gamma(b_i, Y_i)$.) Since (X_i, Y_i) and (Y_i, X_{i+1}) are ε-regular pairs with density at least d, such two vertices c_i and d_i exist. Clearly, $\{a_i, b_i, c_{i-1}, d_{i-1}\}$ induces a $K_{2,2}$ for each $i = 1, 2, \ldots, k$.

To show the main idea on how to find an ESHC in G without going through tedious details, we first consider the ideal case: $V_0 = \emptyset$ and all (X_i, Y_i) are $(\varepsilon, d/w)$-super-regular. For each i let $X_i' = X_i - \{a_i, b_i\}$ and

$Y_i' = Y_i - \{c_i, d_i\}$. It is easy to see that (X_i', Y_i') is also super-regular. For each $1 \leq i \leq k$, we apply the Blow-up Lemma (Lemma 2.2) to (X_i', Y_i') to obtain an ESP

$$y_1, x_1, \ldots, y_{N-2}, x_{N-2}$$

such that

$$y_1 \in \Gamma(\{a_i, b_i\}, Y_i'), x_1 \in \Gamma(d_{i-1}, X_i'), y_2 \in \Gamma(b_i, Y_i'),$$

$$x_{N-3} \in \Gamma(c_i, X_i'), y_{N-2} \in \Gamma(a_{i+1}, Y_i), x_{N-2} \in \Gamma(\{c_i, d_i\}, X_i).$$

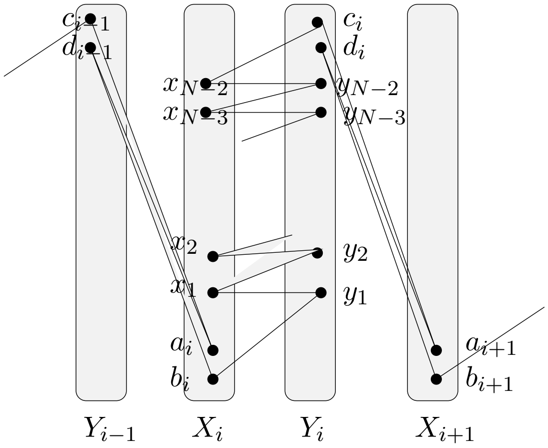


Figure 2. ESHP between X_i and Y_i

By concatenating paths $y_1 x_1 y_2 x_2 \ldots y_{N-2} x_{N-2} c_i a_{i+1} d_i b_{i+1}$ for $i = 1, 2, \ldots, \ell$, we obtain an ESHC of G.

For the general case, we divide the proof into the following three steps.

Step 1: For each $i \geq 1$, we will remove vertices from $X_i \cup Y_i$ to V_0 such that the remaining graph having minimum degree at least $(d - 2\varepsilon)N$.

Step 2: We will divide the vertices in V_0 into pairs (x, y) and associate each pair of vertices with either a pair or two pairs of clusters.

Step 3: We will apply the Blow-up Lemma to each remaining pair (X_i, Y_i) to form an ESHC, where the associated vertices will be included.

We now give the details.

Step 1. For each $i \geq 1$, let

$$\begin{aligned} X_i' &:= \{x \in X_i,\ \deg(x, Y_i) \geq (d-\varepsilon)N\} \quad \text{and} \\ Y_i' &:= \{y \in Y_i,\ \deg(y, X_i) \geq (d-\varepsilon)N\}. \end{aligned}$$

If $|X_i'| \neq |Y_i'|$, say $|X_i'| > |Y_i'|$, we pick an arbitrary subset of X_i' of $|Y_i'|$ vertices and still name it X_i'. Let $V_0' := V_0 \bigcup_{i=1}^k (X_i - X_i') \cup (Y_i - Y_i')$, that is, we move all $x \in X_i$ such that $\deg(x, Y_i) < (d-\varepsilon)N$ and all $y \in Y_i$ such that $\deg(y, X_i) < (d-\varepsilon)N$ to V_0. For each i, since (X_i, Y_i) is ε-regular, (X_i', Y_i') is ε-regular and $|X_i'| = |Y_i'| \geq (1-\varepsilon)N$, which in turn shows that $|V_0| \leq 3\varepsilon n$. In addition, the minimum degree $\delta(G'[X_i', Y_i']) \geq (d-\varepsilon)N \geq (d/2)N$.

Step 2. For a cluster $A_i \in \{X_i, Y_i\}$ and a vertex $x \in V$, we say x is *associated* with A_i, denoted by $x \sim A_i$, if $\deg(x, A_i) \geq dN$. For each pair clusters $\{A_i, B_i\} = \{X_i, Y_i\}$, we call a pair vertices (a, b) *associated* with (A_i, B_i), denoted by $(a, b) \sim (A_i, B_i)$, if $a \sim A_i$ and $b \sim B_i$. We call two clusters A_i and A_j *associated* if (A_i, A_j) is an ε-regular with density $d(A_i, A_j) \geq d$. We call two vertices (a, b) *semi-associated* with two pairs of partners $(A_i, B_i; A_j, B_j)$ if $a \sim A_i$, $B_i \sim A_j$, and $b \sim B_j$. By the regularity property, we have the following result.

Claim 4.4. *If (a, b) is semi-associated with $(A_i, B_i; A_j, B_j)$, then there are $(1-\varepsilon)N$ vertices $a_i \in A_i$ and $(1-\varepsilon)N$ vertices $b_i \in B_i$ such that $(a, a_i) \sim (A_i, B_i)$ and $(b_i, b) \sim (A_j, B_j)$.*

Claim 4.5. *Each vertex $v \in V$ is associated with at least $(\frac{1}{2} - 2\beta)\ell$ clusters.*

Proof. Suppose the contrary: *there is a vertex v which is associated with at most $(\frac{1}{2} - 2\beta)\ell$ clusters.* Then,

$$\left(\frac{1}{2} - \beta\right) n \leq \deg_G(v) \leq \left(\frac{1}{2} - 2\beta\right) \ell N + dN\ell + 3\varepsilon n < \left(\frac{1}{2} - \frac{3}{2}\beta\right) n,$$

a contradiction. □

Claim 4.6. *V_0 can be partitioned into disjoint vertex pairs such that each pair (a, b) is $(a, b) \sim (A_i, B_i)$ for some pair of partners (A_i, B_i) or is semi-associated with $(A_i, B_i; A_j, B_j)$ for two pairs of partners (A_i, B_i) and (A_j, B_j). Moreover, each pair of partner is associated with at most $(d/20)N$ such vertex pairs of vertices in V_0.*

Proof. We progressively take a pair of vertices and associate them with a pair of partners. Let $W_0 \subseteq V_0$ have a desired partition associating with pairs of partners fulfilling the above properties. If $W_0 = V_0$ we are done. Suppose $W_0 \neq V_0$. Let $u, w \in V_0 - W_0$ be two distinct vertices.

Let Ω be a set of pairs of partners such that each partner (X_i, Y_i) is associated with $(d/20)N$ pairs of vertices of W_0. Let $\omega = |\Omega|$. Then,

$2(d/20)N\omega \leq |W_0| \leq 3\varepsilon n \leq \varepsilon(6+\varepsilon)kN$, consequently, $\omega \leq \frac{10(6+\varepsilon)\varepsilon}{d}k \leq \beta\ell$. We may assume that (u,w) is not associated with any partner pairs not in Ω, otherwise, we are done.

Let $\mathcal{U}$ be the set of clusters not in Ω associated with u, $\mathcal{W}$ be the et of clusters not in Ω associated with w, and $P(\mathcal{U})$ and $P(\mathcal{W})$ be the set of partners of clusters in $\mathcal{U}$ and $\mathcal{W}$, respectively. Then, we have

$$\mathcal{U} \cap P(\mathcal{W}) = \emptyset,\ P(\mathcal{U}) \cap \mathcal{W} = \emptyset.$$

and

$$E_{G_r}(P(\mathcal{U}), P(\mathcal{W})) = \emptyset.$$

Since u is associated with at least $(1-2\beta)\ell$ clusters, $|\mathcal{U}| \geq (1-3\beta)\ell$. Similarly, $|\mathcal{W}| \geq (1-3\beta)\ell$.

If $\mathcal{U} \cap \mathcal{W} = \emptyset$, then $P(\mathcal{U}) \cap P(\mathcal{W}) = \emptyset$. Since $|P(\mathcal{U})| \geq (\frac{1}{2}-3\beta)\ell$ and $|P(\mathcal{W})| \geq (\frac{1}{2}-3\beta)\ell$, G is in the Extremal Case 2 by Claim 4.2, a contradiction. Let $A \in \mathcal{U} \cap \mathcal{W}$. Then $P(A)$ (the partner of A) is not adjacent to any cluster in $P(\mathcal{U}) \cup P(\mathcal{W})$. Since $\delta(G_r) \geq (\frac{1}{2}-3\beta)\ell$, then $|P(\mathcal{U}) \cup P(\mathcal{W})| \leq (\frac{1}{2}+3\beta)\ell$. Since $|P(\mathcal{U})| \geq (\frac{1}{2}-3\beta)\ell$ and $|P(\mathcal{W})| \geq (\frac{1}{2}-3\beta)\ell$, then $|P(\mathcal{U}) \cap P(\mathcal{W})| \geq (\frac{1}{2}-6\beta)\ell$. If there is an edge in G_r connecting two clusters in $P(\mathcal{U}) \cap P(\mathcal{W})$, we are done. So we may assume that $P(\mathcal{U}) \cap P(\mathcal{W})$ is an independent set in G_r, which in turn shows that G is in the Extremal Case 1, a contradiction. $\square$

Suppose (u,w) is semi-associated with $(A_i, B_i; A_j, B_j)$. Since (A_i, B_i) is an ε-regular pair with density $d(A_i, B_i) \geq d$, there exist at least $(1-\varepsilon)N$ vertices $a_i \in A_i$ such that $\deg(a_i, B_i) \geq dN$ so that $(a_i, u) \sim (A_i, B_i)$; Since (B_i, A_j) is an ε-regular pair with $d(B_i, A_j) \geq d$, there exist at least $(1-\varepsilon)N$ vertices $b_i \in B_i$ such that $\deg(b_i, A_j) \geq dN$ so that $(b_i, w) \sim (A_j, B_j)$. Then, following Claim 4.6, we claim the following result.

Claim 4.7. *There exists a $V_0^* \supseteq V_0$ such that it can be partitioned into k pairs of sets (U_i, W_i) $(1 \leq i \leq k)$ satisfying the following properties.*

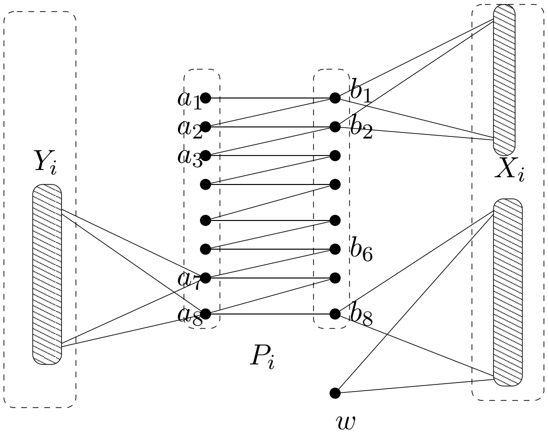
1. *Every $u \in U_i$ is associated with Y_i and every $w \in W_i$ is associated with X_i. Consequently, every pair $(w,u) \in (W_i, U_i)$ is associated with (X_i, Y_i).*
2. *$|V_0^*| \leq 2|V_0| \leq 4\varepsilon n$ and $|U_i| = |W_i| \leq (d/20)N$ for each $i = 1, 2, \ldots, k$.*
3. *For each $1 \leq i \leq k$, $|X_i'| = |Y_i'| \geq (1-d/10)N$, where $X_i' := X_i - V_0^*$ and $Y_i' := Y_i - V_0^*$.*

Claim 4.8. *For each $1 \le i \le k$ and each pair $(U_i^*, W_i^*) \subseteq (U_i, W_i)$ with $|U_i^*| = |W_i^*|$ and each $w \in W_i - W_i^*$, (if $W_i \neq W_i^*$) there is an ESP*

$$P_i = b_1\ a_1\ b_2\ a_2\ b_3\ a_3 \ldots\ a_p\ b_p$$

of $2t_i$ vertices such that the following statements hold.

(1) *$U_i^* \subseteq \{a_1, a_2, \ldots, a_{t_i}\} \subseteq U_i^* \cup X_i'$ and $W_i^* \subseteq \{b_1, b_2, \ldots, b_{t_i}\} \subseteq W_i^* \cup Y_i'$. More specifically, $W_i^* = \{b_1, b_6, b_{11}, \ldots, b_{5j+1}, \ldots\}$ and $U_i^* = \{a_3, a_8, \ldots, a_{5j+3}, \ldots\}$.*
(2) *$p = 5|U_i^*| - 2 = 5|W_i^*| - 2 \le (d/4)N$. Consequently, $|X_i^*| = |Y_i^*| \ge (1 - d/2)N$.*
(3) *$\deg(\{b_1, b_2\}, X_i) \ge (d - \varepsilon)dN$ and $\deg(\{a_{p-1}, a_p\}, Y_i) \ge (d - \varepsilon)dN$.*
(4) *If $(U_i^*, W_i^*) \neq (U_i, W_i)$, there exist $w \in W_i - W_i^*$ such that $\deg(\{b_p, w\}, X_i) \ge (d - \varepsilon)dN$.*



$a_3, a_8 \in U_i^*$ and $b_1, b_6 \in W_i^*$

Figure 3. An example of $|U_i^*| = |V_i^*| = 2$

Proof. Let $u_i \in U_i^*$ and $w_i \in W_i^*$. Suppose there is a desired EPS $P_i := b_1\ a_1\ b_2\ a_2 \ldots b_p\ a_t$ for the triple $(U_i^* - u_i, W_i^* - w_i, w_i)$. Let $T \subseteq \Gamma(u_i, Y_i)$ such that $\deg(v, \Gamma(w, X_i)) \ge (d - \varepsilon)N$. Since $\deg(w, X_i) \ge dN$ and (X_i, Y_i) is ε-regular with density $d(X_i, Y_i) \ge d$, we have $|T| \ge (d - \varepsilon)\deg(u_i, Y_i) \ge (d - \varepsilon)dN$. Let $S := \Gamma(\{b_t, w_i\}, X_i)$ and $R := \Gamma(\{a_{t-1}, a_t\}, Y_i)$. From our assumption, we have $|S| \ge (d-\varepsilon)dN$ and $|R| \ge (d-\varepsilon)dN$. By the regularity, S contains a subset S_1 such that $\deg(v, \Gamma(u_i, Y_i)) \ge (d - \varepsilon)\deg(u_i, Y_i) \ge (d-\varepsilon)dN$. Using the regularity again, there is a vertex $s_1 \in S - (V_0^* \cup V(P_i))$ such that $\deg(s, R) \ge (d - \varepsilon)|R|$ and $\deg(s, T) \ge (d - \varepsilon)|T|$. Continuing in this argument, we find four vertices $s_1, s_2, s_3, s_4 \in S$, two vertices $r_1, r_2 \in R$

and two vertices $t_1, t_2 \in T$ such that the edges between $\{s_1, s_2, s_3, s_4\}$ and $\{r_1, r_2, t_1, t_2\}$ form $K_{4,4}$. Then

$$P_i\ r_1\ s_1\ r_2\ s_2\ w_i\ s_3\ t_1\ s_4\ t_2\ u_i$$

is a desired EPS for the triple (U_i^*, W_i^*, w). $\square$

For each i, let $P_i = b_{i1}a_{i1}b_{i2}a_{i2}\dots b_{ip_i}a_{ip_i}$ be a desired ESP for the pair (U_i, W_i), $X_i^* = X_i' - V(P_i)$ and $Y_i^* = Y_i' - V(P_i)$. Since $|V(P_i)| = 5|U_i \cup W_i| - 4 \le (d/4)N$, $|X_i^*| = |Y_i^*| \ge (1-(1/10)d-(1/5)d)N \ge (1-d/4)N$. Consequently, each (X_i^*, Y_i^*) pair is ε-regular with density $d(X_i^*, Y_i^*) \ge d/2$.

Step 3. The following simple fact is needed before we apply the blow-up lemma.

Claim 4.9. *Suppose $C = x_1y_1x_2y_2x_3y_3\dots$ induces an ESC and $P = b_1a_1b_2a_2\dots b_pa_p$ induces an ESP. If*

$$x_1 \in \Gamma(b_1), \quad x_2 \in \Gamma(b_1) \cap \Gamma(b_2), \quad y_1 \in \Gamma(a_1)$$
$$x_3 \in \Gamma(b_p), \quad y_2 \in \Gamma(a_{p-1}) \cap \Gamma(a_p), \quad y_3 \in \Gamma(a_p)$$

then $x_1y_1x_2Py_2Cx_1$ is an ESC.

We now apply the Blow-up Lemma to each (X_i^*, Y_i^*) in the same way as the idea case to obtain an ESHC in $G - \cup_{i=1}^k V(P_i)$ with specifying six consecutive vertices, say x_{11}, y_{11}, x_{12}, y_{12}, x_{13} and y_{13} such that

$$x_{11} \in \Gamma(b_{i1}), \quad x_{12} \in \Gamma(b_{i1}) \cap \Gamma(b_{i2}), \quad y_{11} \in \Gamma(a_{i1})$$
$$x_{13} \in \Gamma(b_{ip_i}), \quad y_{12} \in \Gamma(a_{i(i_p-1)}) \cap \Gamma(a_{ip_i}), \quad y_{13} \in \Gamma(a_{ip_i}).$$

Using Claim 4.9, we insert each P_i into the ESHC to get an ESHC of G to complete the proof of the Non-extremal Case. $\square$

4.2. Extremal Cases.

In this section we prove the main theorem under two extremal cases.

4.2.1. *Extremal Case 1.* The following theorem covers this case.

Theorem 4.10. *Suppose that $G = (V, E)$ is a graph oh n vertices with $\delta(G) \ge \frac{n}{2} + 3$. If there exists a balanced partition of V into V_1 and V_2 such that $d(V_1, V_2) \ge 1 - \alpha$ for some $\alpha \ll 1$, then G contains an ESHC.*

Proof. We start with defining the sets of *typical* vertices in V_1 and V_2. Let $\alpha_1 = \alpha^{1/3}$ and $\alpha_2 = \alpha^{2/3}$. For each $i = 1, 2$, we define

$$V_i' = \{x \in V : \deg(x, V_{3-i}) \ge (1 - \alpha_1)\frac{n}{2}\}.$$

Since $d(V_1, V_2) \geq 1 - \alpha$, we have $|V_i - V_i'| \leq \alpha_2 n/2$ and consequently $|V_i'| \geq (1-\alpha_2)n/2$ for $i = 1, 2$. Let $V_0 = V - V_1' - V_2'$. The subgraph $G[V_1', V_2']$ is almost complete: for any $x \in V_i'$,

$$\deg(x, V_{3-i}') > (1-\alpha_1)\frac{n}{2} - \alpha_2\frac{n}{2}. \tag{1}$$

We consider two cases.

Case 1: $|V_1'|, |V_2'| \leq n/2$.

Case 1a: $V_0 = \emptyset$. Then $|V_1'| = |V_2'| = n/2$. Suppose that $n/2 = 3N + r$ for some integers N and $r \in \{0, 1, 2\}$. If $r = 0$, then arbitrarily partition V_1' into $X_1, \ldots, X_N \subset V_1'$ and V_2' into $Y_1, \ldots, Y_N \subset V_2'$ such that all X_i, Y_i are 3-vertex sets. Next define a new bipartite graph H whose vertices are X_i, Y_i for i, and X_i is adjacent to Y_j if and only if every vertex in X_i is adjacent all the vertices in Y_j, *i.e.*, $G[X_i, Y_j]$ is complete. The minimum degree of H satisfies

$$\delta(H) \geq N - 3\alpha_1\frac{n}{2} \geq \frac{N}{2}$$

whenever $\alpha_1 \leq \frac{1}{18}$ or $\alpha \leq (1/18)^3$. Applying a well-known result of Moon-Morse [**23**]: *In a bipartite graph H with p vertices in each color class, if $\delta(H) > p/2$, then H contains a Hamilton cycle*, we obtain a Hamilton cycle in H. Assume the Hamilton cycle is $X_1Y_1 \ldots, X_NY_N$ with $X_i = \{x_i^1, x_i^2, x_i^3\}$ and $Y_i = \{y_i^1, y_i^2, y_i^3\}$. Then

$$x_1^1\ y_N^3\ x_1^2\ y_1^1 x_1^3\ y_1^2\ x_2^1\ y_1^3\ x_2^2\ y_2^1 \ldots x_N^3\ y_N^2$$

is an ESHC of G.

If $n/2 = 3N + 1$, then we need to remove an edge a_1b_1 with $a_1 \in V_1'$ and $b_1 \in V_2'$ first; if $n/2 = 3N + 2$, then we remove a copy of $K_{2,2}$ with $a_1, a_2 \in V_1'$ and $b_1, b_2 \in V_2'$ (this is possible because $G[V_1', V_2']$ is dense). Next, as shown above, we find an ESC $C = x_1, y_1, \ldots, x_m, y_m$ covering the remaining vertices of $V_1' \cup V_2'$, where $x_i \in V_1'$ and $y_i \in V_2'$. Finally we add a_1b_1 or $a_1b_1a_2b_2$ into C obtaining a larger ESC by the following proposition.

Proposition 4.11. *Suppose that $P = a_1b_1 \ldots a_kb_k$ is an ESP on $2k$ vertices for some $k \geq 1$, and $C = x_1, y_1, \ldots, x_m, y_m$ is an ESC on $2m$ vertices for some $m \geq 4$. Let $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_m\}$. If $\deg(a_i, Y) > 7m/8$ for $i = 1, 2$ and k and $\deg(b_i, Y) > 7m/8$ for $i = 1, k-1$ and k, then we can insert P into C to get a new ESC*

$$x_1y_1 \ldots x_ib_ka_k \ldots b_1a_1y_i \ldots x_my_m.$$

PROOF. *Case 1:* $k = 1$.

In order to have the desired ESC, we need adjacency

$$b_1 \sim x_{i-1}, x_i, x_{i+1}, \quad a_1 \sim y_{i-1}, y_i, y_{i+1} \tag{2}$$

for some i. Since $\deg(b_1, X) > 7m/8$, we have

$$|\{i : b_1 \not\sim x_i\}| < \frac{m}{8}, \quad |\{i : b_1 \not\sim x_{i-1}\}| < \frac{m}{8}, \quad |\{i : b_1 \not\sim x_{i+1}\}| < \frac{m}{8}$$

(assuming that $x_0 = x_m$ and $x_{m+1} = x_1$). Similarly we have

$$|\{i : a_1 \not\sim y_i\}| < \frac{m}{8}, \quad |\{i : a_1 \not\sim y_{i-1}\}| < \frac{m}{8}, \quad |\{i : a_1 \not\sim y_{i+1}\}| < \frac{m}{8}.$$

Therefore the number of indices i satisfying (2) is greater than $m - 6m/8 > 0$.

Case 2: $k \geq 2$.

In order to have the desired ESC, we need adjacency

$$b_k \sim x_{i-1}, x_i, \; a_k \sim y_{i-1}, \; b_{k-1} \sim x_i, \; a_1 \sim y_i, y_{i+1}, \; b_1 \sim x_{i+1}, \; a_2 \sim y_i \tag{3}$$

for some i. Using the degree condition, similar as in the $k = 1$ case, we conclude that the number of indices i satisfying (3) is greater than $m - 8m/8 = 0$. □

It is easy to check that the condition of Proposition 4.11 is met: since a_1, b_1 (and a_2, b_2) are typical vertices, we have $\deg(a_i, V_2') > (1 - \alpha_1)n/2 > (7/8)(n/2)$, and $\frac{n}{2} - 2 \leq m \leq \frac{n}{2} - 1$.

Case 1b: $V_0 \neq \emptyset$. For each $v \in V_0$, we have $\deg(v, V_1), \deg(v, V_2) \leq (1 - \alpha_1)\frac{n}{2}$, which implies that $\deg(v, V_2), \deg(v, V_1) \geq \alpha_1 \frac{n}{2}$ and

$$\deg(v, V_i') \geq (\alpha_1 - \alpha_2)\frac{n}{2} \quad \text{for } i = 1, 2. \tag{4}$$

We partition V_0 into two sets W_1 and W_2 such that $|W_1| = n/2 - |V_1'|$ and $|W_2| = n/2 - |V_2'|$. Let $t_1 = |W_1|$ and $t_2 = |W_2|$. Then $t_1, t_2 \leq \alpha_2 n/2$. For each $v \in W_1$, we greedily find four different neighbors in V_2' (*i.e.*, the sets of 4 neighbors are disjoint). This can be done because (4) and

$$(\alpha_1 - \alpha_2)\tfrac{n}{2} \geq 4\alpha_2 \tfrac{n}{2} \geq 4t_1.$$

Next we find $3t_1$ vertices from V_1' to form an EPS together with these 4-stars as follows. Suppose $x_1, x_2, \ldots, x_s$ is a linear order of vertices of W_1 and $a_i, b_i, c_i, d_i \in V_2'$ are the selected four neighbors of x_i. We choose distinct vertices $u_i, v_i, w_i \in V_1'$, $1 \leq i \leq t_1$, such that

$$u_i \sim a_i, b_i, c_{i-1}, d_{i-1}, \quad v_i \sim a_i, b_i, c_i, d_{i-1}, \quad w_i \sim a_{i+1}, b_{i+1}, c_i, d_i$$

(ignore a vertex if its index is 0 or t_1+1). This is possible because each a_i, b_i, c_i, d_i has at least $(1-\alpha_1-\alpha_2)n/2$ neighbors in V_1 and

$$(1-\alpha_1-\alpha_2)\frac{n}{2} \geq 3\alpha_2\frac{n}{2} \geq 3t_1.$$

We thus obtain an ESP $P_1 = \{u_i a_i v_i b_i x_i c_i w_i d_i\}_{i=1}^{t_1}$. Similarly we construct t_2 an ESP P_2 of length $4t_2$ containing all the vertices of W_2. Next find 4 vertices $x_1, x_2 \in V_1'$, $y_1, y_2 \in V_2'$ such that $P = P_1 x_1 y_1 x_2 y_2 P_2$ is an ESP on $8t_1+8t_2+4$ vertices. Denote by V_1'' and V_2'' the sets of the remaining vertices in V_1' and V_2', respectively. Then $|V_i''| = \frac{n}{2} - (4t_1+4t_2+2)$ for $i=1,2$. Now we proceed as in Case 1a: remove at most two vertices from each of V_1'' and V_2'' and add them to P; find an ESC C covering the remaining vertices of V_1'' and V_2''; finally apply Proposition 4.11 to join P and C obtaining an ESHC of G. The condition of Proposition 4.11 is met because the first and last three vertices in P are typical vertices and

$$\frac{n}{2} - (4t_1+4t_2+4) > 8(\alpha_1+\alpha_2)\frac{n}{2}.$$

Case 2: One of $|V_1'|, |V_2'|$ is greater than $n/2$.

Suppose that $|V_1'| > n/2$. We first move all vertices in V_0 to V_2'; and if there is a vertex $v \in V_1'$ such that $\deg(v, V_1') \geq \alpha_1 n/2$, then we move v to V_2' as well. Continue this till either $|V_1'| = n/2$ or no more such vertex exists. If $|V_1'| = n/2$, then we proceed as in Case 1. Otherwise assume $|V_1'| = \frac{n}{2} + t_1$ for some $t_1 > 0$. We have $\deg(v, V_1') < \alpha_1 n/2$ for every $v \in V_1'$. The current V_2' contains a set W_2 of new vertices that were moved from V_0 or V_1'. We have $|W_2| + t_1 \leq \alpha_2 n/2$ and $\deg(v, V_1') \geq (\alpha_1 - \alpha_2)n/2$ for every $v \in W_2$.

Recall that $\delta(G) \geq n/2+3$. The induced graph $G[V_1']$ has minimum degree

$$\delta = \delta(G) - |V_2'| \geq \left(\frac{n}{2}+3\right) - \left(\frac{n}{2} - t_1\right) \geq t_1 + 3$$

and maximum degree $\Delta \leq \alpha_1 n/2$. Applying Lemma 3.7, there are at least

$$\frac{t_1|V_1'|}{5(\alpha_1\frac{n}{2}+t_1)} \geq \frac{t_1\frac{n}{2}}{5(\alpha_1+\alpha_2)\frac{n}{2}} \geq t_1$$

disjoint 4-starts in $G[V_1']$. Pick t_1 such 4-stars and move their centers to V_2'. Next construct disjoint 4-stars in $G[V_1', W_2]$ covering all the vertices of W_2 such that the new 4-stars are also disjoint from the existing 4-stars. We now proceed as in Case 1b.

$\square$

4.3. Extremal Case 2.

The following theorem covers this case.

Theorem 4.12. *Suppose that $G = (V, E)$ is a graph oh n vertices with $\delta(G) \geq \frac{n}{2} + 306$. If there exists a balanced partition of V into V_1 and V_2 such that $d(V_1, V_2) \leq \alpha$ for some $\alpha \ll 1$, then G contains an ESHC.*

Proof. We start with defining the sets of *typical* vertices in V_1 and V_2. Let $\alpha_1 = \alpha^{1/3}$ and $\alpha_2 = \alpha^{2/3}$. We define

$$V_i' = \{x \in V : \deg(x, V_{3-i})\} < \alpha_1 \frac{n}{2}.$$

for $i = 1, 2$. Since $\delta(G) > n/2$, we have $\deg(x, V_i) > (1-\alpha_1)n/2$ for every $x \in V_i'$. Since $d(V_1, V_2) \leq \alpha$, we have $|V_i - V_i'| \leq \alpha_2 n/2$ and $|V_i'| \geq (1-\alpha_2)n/2$ for $i = 1, 2$. Consequently for every $x \in V_i'$,

$$\deg(x, V_i') > |V_i'| - \alpha_1 \frac{n}{2} \geq (1 - \alpha_1 - \alpha_2)\frac{n}{2}. \tag{5}$$

Let $V_0 = V - V_1' - V_2'$ and call the vertices in V_0 *atypical.* Then $|V_0| \leq \alpha_2 n$ and $\deg(x, V_i') \geq (\alpha_1 - \alpha_2)n/2$ for all $x \in V_0$.

The following two steps together provide an ESHC of G.

Step 1. Find two disjoint ESP's $x_1 \dots x_p$ and $y_1 \dots y_p$ of length $p \leq 14$ such that $x_1, x_2, x_3, y_1, y_2, y_3 \in V_1'$ and $x_{p-2}, x_{p-1}, x_p, y_{p-2}, y_{p-1}, y_p \in V_2'$.

Step 2. Find two disjoint ESP's P_1 and P_2 containing all uncovered vertices in V_1' and V_2', respectively, such that

$$P_1 = x_3\ x_2\ x_1 \dots y_1\ y_2\ y_3, \quad P_2 = x_{p-2}\ x_{p-1}\ x_p \dots y_p\ y_{p-1}\ y_{p-2}.$$

While Step 2 is routine and handled similarly as in Extremal Case 1, Step 1 is not so easy – it costs us the large constant 306 in the min-degree condition.

4.3.1. *Step 1: Connect V_1' and V_2'.* We need to find two disjoint ESP's $x_1 \dots x_p$ and $y_1 \dots y_p$ of length $p \leq 14$ such that $x_1, x_2, x_3, y_1, y_2, y_3 \in V_1'$ and $x_{p-2}, x_{p-1}, x_p, y_{p-2}, y_{p-1}, y_p \in V_2'$. We call these two ESP's *connectors connecting V_1' and V_2'*, or simply *connectors.* The simplest connector is an ESP $x_1 \dots x_6$ with $x_1, x_2, x_3 \in V_1'$ and $x_4, x_5, x_6 \in V_2'$. Unfortunately such a simple connector may not exist because $e(V_1', V_2')$ can be very small. Define

$$V_0' = \{x \in V_0 : \deg(x, V_i') \geq n/7 \text{ for } i = 1, 2\}.$$

If $|V_0'| > 297$, then we immediately obtain two vertex-disjoint connectors, each of which contains 3 vertices of V_0' by the following claim.

Claim 4.13. *If $|V_0'| > 294$, then there are subsets $X \subset V_1'$, $Y \subset V_2'$ and $Z \subset V_0'$ such that $|X| = |Y| = 2$, $|Z| = 3$, and $G[X, Z] \cong H[Y, Z] \cong K_{2,3}$.*

Proof. We apply the standard counting arguments on the existence of $K_{2,t}$ in a bipartite graph $H[A,B]$. If H contains no $K_{2,t}$ with t vertices in A and 2 vertices in B, then

$$\sum_{x\in A}\binom{\deg(x,B)}{2}\le (t-1)\binom{|B|}{2} \tag{6}$$

Let $A=V_0'$ and $B=V_1'$, and note that $|B|\le n/2$ and $\deg(x,B)\ge n/7$ for every $x\in A$. If (6) holds, then $|V_0'|\le 49(t-1)/4$ as $n\to\infty$. Let $t=25$. So when $|V_0'|>294$, there exists a copy of $K_{2,t}$ on $Z'\subseteq V_0'$ of size t and $X\subset V_1'$ of size 2. Now we repeat these arguments to $G[Z',V_2']$. Since $t=25>49(3-1)/4$ and obtain a copy of $K_{2,t}$ on $Z\subseteq C'$ of size 3 and $Y\subset V_2'$ of size 2. □

When $|V_0'|>294+3$, we apply Claim 4.13 twice to obtain two disjoint subsets $X^1,X^2\subset V_1'$, $Y^1,Y^2\subset V_2'$ and $Z^1,Z^2\subset V_0'$ such that $|X^i|=|Y^i|=2$, $|Z^i|=3$, and $G[X^i,Z^i]\cong H[Y^i,Z^i]\cong K_{2,3}$ for $i=1,2$. To see why this gives two desired ESP's, assume $X^1=\{x_3,x_5\}$, $Y^1=\{x_7,x_9\}$, and $Z^1=\{x_4,x_6,x_8\}$. Then $x_3,x_4,\dots,x_8,x_9$ is an ESP. Since $x_4\notin V_1'$ and $x_8\notin V_2'$, we extend this ESP by adding two vertices from V_1' in the beginning and two vertices from V_2' at the end. Since $x_3,x_5\in V_1'$, by (5), we easily find $x_2\in\Gamma(x_3,x_5,V_1')$. Since $\deg(x_2,V_1')>|V_1'|-\alpha_1 n/2$ and $\deg(x_4,V_1')>n/7$, we can find $x_1\in\Gamma(x_2,x_4,V_1')$. Therefore $x_1,x_2,\dots,x_9$ is an ESP. Similarly we find $x_{10},x_{11}\in V_2'$ such that $x_1,x_2,\dots,x_{10},x_{11}$ is an ESP. We repeat this to find the other ESP.

Now assume that $c_0=|V_0'|\le 297$. We will call the vertices in V_0' *forbidden* vertices because they will *not* be used in either connector. Since $|V_0|\le\alpha_2 n$, all vertices $x\in V_0$ satisfy $\deg(x,V_1'\cup V_2')>n/2-\alpha_2 n$. If $x\in V_0-V_0'$, then exactly one of $\deg(x,V_1)$ and $\deg(x,V_2)$ is less than $n/7$. We thus partition V_0-V_0' into W_1 and W_2 with

$$W_i=\{x\in V_0-V_0':\deg(x,V_{3-i}')<n/7\}.$$

For each $x\in W_i$, we have

$$\deg(x,V_i')\ge\deg(x,V_1'\cup V_2')-\frac{n}{7}\ge\frac{n}{2}-\alpha_2 n-\frac{n}{7}=\frac{5}{14}n-\alpha_2 n. \tag{7}$$

Let $U_i=V_i'\cup W_i$ for $i=1,2$. Without loss of generality, assume that $|U_1|=n/2-b$ for some integer $b\ge 0$. Then $|U_2|=n/2+b-c_0$. Since $|V_1'|\ge(1-\alpha_2)n/2$, then $|V_2'|\le(1+\alpha_2)n/2$. By (5) and (7), all vertices $x\in U_i$ satisfy

$$\deg(x,V_i')\ge\frac{5}{14}n-\alpha_2 n. \tag{8}$$

Our goal is to find two disjoint ESP's connecting U_1 and U_2: $u_1\dots u_q$ and $v_1\dots u_q$ with $q\le 8$ such that $u_i,v_i\in U_1$ for $i=1,2,3$, and $u_i,v_i\in$

U_2 for $i = q-2, q-1, q$. If any of u_1, u_2, u_3 is not from V_1', then we find at most three new vertices $x_1, x_2, x_3 \in V_1'$ such that $x_1x_2x_3u_1u_2u_3$ is an ESP. For example, assume that $u_1 \notin V_1'$. Then we first find $x_3 \in \Gamma(u_1, u_3, V_1')$, then $x_2 \in \Gamma(x_3, u_2, V_1')$, and finally $x_1 \in \Gamma(x_2, u_1, V_1')$ by applying (8) three times. Similar we add at most three vertices from V_2' $x_{q+1}, x_{q+2}, x_{q+3}$ such that $u_{p-2}, u_{p-1}, u_p, x_{q+1}, x_{q+2}, x_{q+3}$ is an ESP. The ESP $x_1x_2x_3u_1 \dots u_qx_{q+1}, x_{q+2}, x_{q+3}$ is one desired connector of V_1' and V_2'. We obtain another connector similarly.

We now show how to obtain an ESP connecting U_1 and U_2. Recall that $\delta(G) \geq n/2 + k$ with $k = c_0 + 9$. Then for every $v \in U_1$,

$$\deg(v, U_2) \geq \frac{n}{2} + k - \left(\frac{n}{2} - b\right) - c_0 = k + b - c_0 > 0. \tag{9}$$

However, since b may be large, we do not have such a min-degree condition for *all* vertices in U_2. Define $U_2^* = \{y \in U_2 : y \sim x \text{ for some } x \in V_1'\}$. Note that $U_2^* \neq \emptyset$ because every vertex in V_1' has some neighbor in U_2. Suppose that y_0 has the maximum degree in U_2 among all vertices in U_2^*. Pick any vertex $x_2 \in \Gamma(y_0, V_1')$. Let $B_1 = \Gamma(x_2, V_1')$ and $B_2 = \Gamma(y_0, U_2)$. Then $|B_1| \geq (1 - \alpha_1 - \alpha_2)n/2$ by (5) and $|B_2| \geq 5n/14 - \alpha_2 n$ by (8). If there exists a 3-path $x_1x_3x_4x_6$ in $G[B_1, B_2]$, then $x_1x_2x_3x_4y_0x_6$ is the desired ESP (here x_1, x_2, x_3 are already in V_1' so we do not need to add any vertex before them).

We may therefore assume that $G[B_1, B_2]$ contains no 3-path. This implies the bipartite graph $G[B_1, B_2]$ consists of disjoint stars, in particular, $e(B_1, B_2) < |B_1| + |B_2|$. Let $B_1' = \{x \in B_1 : \deg(x, B_2) \leq 1\}$. The vertices in $B_1 - B_1'$ have disjoint neighborhoods of size at least 2 in B_2. Thus $|B_1 - B_1'| \leq |B_2|/2 \leq (1 + \alpha_2)n/4$ and consequently $|B_1'| \geq n/4 - \alpha_1 n$ (in particular $B_1' \neq \emptyset$).

Let $A_2 = U_2 - B_2$ and set $m = |A_2|$. Since $|B_2| \geq 5n/14 - \alpha_2 n$, then $m \leq (1 + \alpha_2)n/2 - (5n/14 - \alpha_2 n) \leq n/7 + 2\alpha_2 n$. For all $v \in B_1'$, by (9),

$$\deg(v, A_2) \geq k + b - c_0 - 1. \tag{10}$$

In particular, $m \geq k + b - c_0 - 1$. On the other hand, for every $u \in U_2^*$, we have $\deg(u, U_2) \leq \deg(y_0, U_2) = n/2 + b - c_0 - m$ by the choice of y_0 and consequently

$$\deg(u, U_1) \geq \frac{n}{2} + k - \left(\frac{n}{2} + b - c_0 - m\right) - c_0 = k - b + m. \tag{11}$$

Since $m \geq k + b - c_0 - 1$, we derive that $\deg(u, U_1) \geq 2k - c_0 - 1 > 0$ for $u \in U_2^*$.

We observe that it suffices to show that $G[U_1, U_2^*]$ contains a copy of $K_{2,2}$. In fact, assume $x_1, x_2 \in U_1$ and $y_1, y_2 \in U_2^*$ are the four vertices of $K_{2,2}$. By

(11), y_1 has a neighbor $x_4 \in U_1 - \{x_1, x_2\}$. By (8), x_1, x_2, x_4 has a common neighbor $x_3 \in V_1'$. Similarly, by (10) and (8), we find $y_4 \in A_2 - \{y_1, y_2\}$ and $y_3 \in \Gamma(y_1, y_2, y_4, V_2')$. This gives an ESP $P = x_4x_3x_2y_1x_1y_2y_3y_4$ connecting U_1 and U_2.

We need some simple facts on the existence of $K_{2,2}$ in bipartite graphs, originally given by Kővári, Sós, and Turán. The proof follows from (6) easily; for completeness, we include it below.

Proposition 4.14. *Let $H = (A, B; E)$ be a bipartite graph such that $|A| = n$, $|B| = m$. Then H contains a copy of $K_{2,2}$ if either of the following holds.*

1. $\deg(x) \geq \sqrt{m}$ *for all* $x \in A$ *and* $n > m + \sqrt{m}$,
2. $e = |E| \geq \max\{3n, m^2/2\}$.

Proof. Suppose instead, H contains no copy of $K_{2,2}$. Then (6) hold with $t = 2$.

Part 1. We have

$$n\binom{\sqrt{m}}{2} \leq \sum_{x \in A} \binom{\deg(x)}{2} \leq \binom{m}{2},$$

which implies that $n \leq \frac{m(m-1)}{\sqrt{m}(\sqrt{m}-1)} = \sqrt{m}(\sqrt{m}+1)$, a contradiction.

Part 2. By convexity, we have $n\binom{e/n}{2} \leq \binom{m}{2}$, which implies $e(e/n - 1) < m^2$. Since $e \geq m^2/2$, we have $(e/n - 1) < 2$ or $e < 3n$, a contradiction. $\square$

We now separate cases by whether $b \geq \sqrt{m}$.

Case 1: $b \geq \sqrt{m}$.

Then (10) gives $\deg(x, A_2) \geq \sqrt{m}$ for any $x \in B_1'$ as long as $k \geq c_0 + 1$. Since $|B_1'| \geq n/4 - \alpha_1 n$ and $m \geq n/7 + 2\alpha_2 n$, we have $|B_1'| > |A_2| + \sqrt{|A_2|}$. Applying Proposition 4.14, we obtain a copy of $K_{2,2}$ from $G[B_1', A_2]$. Note that the two vertices of this $K_{2,2}$ in A_2 belong to U_2^* because they both have neighbors in $B_1' \subseteq V_1'$.

Case 2: $b < \sqrt{m}$. Let $A_2^* = U_2^* \cap A_2$ and set $\ell = |A_2^*|$. By (11), every $u \in A_2^*$ satisfies $\deg(u, U_1) \geq m - \sqrt{m} + k \geq m/2$. This implies $e(A_2^*, U_1) \geq \ell^2/2$. On the other hand, (9) implies that $e(B_1, U_2) \geq 5|B_1|$ as long as $k + b - c_0 \geq 5$. Recall that $e(B_1, B_2) < |B_1| + |B_2|$. By the definition of B_1 and A_2^*,

$$e(B_1, A_2^*) = e(B_1, A_2) > 5|B_1| - (|B_1| + |B_2|) > 4\left(\frac{n}{2} - \alpha_1 n\right) - (1+\alpha_2)\frac{n}{2} > \frac{3n}{2}.$$

Consequently $e(U_1, A_2^*) \ge e(B_1, A_2^*) > 3n/2 \ge 3|U_1|$. Now we apply Proposition 4.14 Part 2 to obtain a copy of $K_{2,2}$ in $G[U_1, A_2^*]$.

After finding one ESP of connecting U_1 and U_2, we move all its vertices to V_0'. As a result, U_1 and U_2 each loses at most 4 vertices. In the worst case, $|V_0'| = c_0' = c_0 + 8$ and $|U_1| = n/2 - b'$ with $b' = b + 4$. We now repeat the entire procedure starting from redefining U_2^*, y_0, etc. Only inequalities that are influenced by new constants are $k \ge c_0' + 1$ and $k + b' - c_0' \ge 5$. Both of them are satisfied with $k = c_0 + 9 = 306$.

4.3.2. *Find two ESP's covering the remaining vertices.* Assume that $P^1 = x_1 \dots x_p$ and $P^2 = y_1 \dots y_p$ are two connectors of length $p \le 14$. Let $S = V(P^1) \cup V(P^2)$, the set of all vertices on P^1 and P^2. Write $U_1 = V_1' - S$ and $U_2 = V_2' \cup V_0 - S$. Let $G_i = G[U_i]$ be the induced subgraph of G on U_i for $i = 1, 2$. Since $U_1 \subseteq V_1'$, by (5), we have $\delta(G_1) \ge |U_1| - \alpha_1 n/2$.

We first show how to find an ESHP P of G_1 such that $x_3x_2x_1Py_1y_2y_3$ is an ESP. Let $n_1 = |V(G_1)|$. The facts that $\delta(G_1) \ge n_1 - \alpha_1 n/2$ and $n_1 \ge (1-\alpha_2)n/2 - 28$ implies that $\delta(G_1) \ge 3n_1/4$. We apply Theorem **??** to obtain a *cube* of Hamilton cycle $v_1, \dots, v_{n_1}$ of G_1. Furthermore, the number of indices i satisfying

$$x_3 \sim v_i, x_2 \sim v_{i-1}, x_1 \sim v_i, v_{i-2}, y_1 \sim v_{i+1}, v_{i+3}, y_2 \sim v_{i+2}, y_3 \sim v_{i+1}$$

is at least $n_1 - 8\alpha_1 n/2 > 0$ because of (5). Let i be such an index and we thus obtain the desired ESP

$$x_3\ x_2\ x_1\ v_i \dots v_1\ v_{n_1} \dots v_{i+1}\ y_1\ y_2\ y_3.$$

Next we show how to find a similar ESHP in G_2. Let $V_2'' = V_2' - S$ and $W_2 = V_0 - S$. Then $U_2 = V_2'' \cup W_2$. We have $n/2 - 28 \le |U_2| \le n/2 + \alpha_2 n$ and $|W_2| \le \alpha_2 n$. For every $x \in V_2''$, $\deg(x, U_2) \ge \deg(x, V_2'') \ge |V_2''| - \alpha_1 n/2$.

We first use the greedy algorithm to find disjoint 4-stars with all the vertices of W_2 as centers and vertices in V_2'' as leaves. This is possible because $\deg(x, V_2'') \ge (\alpha_1 - \alpha_2)\frac{n}{2} - 28 \ge 4|W_2|$ for each $x \in W_2$. Suppose that $W_2 = \{w_1, \dots, w_t\}$ and denote by a_i, b_i, c_i, d_i the four leaves under w_i. We next extend each 4-star into a 7-vertex ESP $P^{(i)} = a_ip_ib_iw_ic_iq_id_i$ by selecting new vertices $p_i \in \Gamma(a_i, b_i, c_i, V_2'')$ and $q_i \in \Gamma(b_i, c_i, d_i, V_2'')$. We can do this for all $i = 1, \dots, t$ because $\deg(\{a_i, b_i, c_i\}, V_2'') \ge |V_2''| - 3\alpha_1 n/2 \ge 6t$. Then we add 3 (different) vertices between $P^{(i)}$ and $P^{(i+1)}$ for $i = 1, \dots, t-1$ to form a single ESP P_0. On the other hand, by the same procedure as in

G_1, we find an ESP $P_2' = x_{p-2}\, x_{p-1}\, x_p\, v_1 \dots v_{n_2}\, y_p\, y_{p-1}\, y_{p-2}$ covering all the remaining vertices in U_2, where $n_2 = |U_2| - 7t_1 - 3(t_1 - 1) \geq \frac{n}{2} - 11\alpha_2 n$. Finally we insert P_0 into P_2' to obtain the desired ESP

$$x_{p-2}\ x_{p-1}\ x_p\ v_1 \dots v_i\ P_0\ v_{i+1} \dots v_{n_1}\ y_p\ y_{p-1}\ y_{p-2}.$$

This insertion is similar to the one given in Proposition 4.11: it is valid because the first three and last three vertices in P_0 are typical and each of them has at least $n_2 - \alpha_1 n/2 > 7n_2/8$ neighbors in $\{v_1, \dots, v_{n_2}\}$.

This complete the proof of Theorem 4.12 and the main theorem. □

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