ON MULTIPARTITE HAJNAL-SZEMERÉDI THEOREMS

JIE HAN AND YI ZHAO

Abstract. Let $G$ be a $k$-partite graph with $n$ vertices in parts such that each vertex is adjacent to at least $\delta^*(G)$ vertices in each of the other parts. Magyar and Martin [20] proved that for $k = 3$, if $\delta^*(G) \geq \frac{2}{3}n$ and $n$ is sufficiently large, then $G$ contains a $K_3$-factor (a spanning subgraph consisting of $n$ vertex-disjoint copies of $K_3$) except that $G$ is one particular graph. Martin and Szemerédi [21] proved that $G$ contains a $K_4$-factor when $\delta^*(G) \geq \frac{3}{4}n$ and $n$ is sufficiently large. Both results were proved by the Regularity Lemma. In this paper we give a proof of these two results by the absorbing method. Our absorbing lemma actually works for all $k \geq 3$.

1. Introduction

Let $H$ be a graph on $h$ vertices, and let $G$ be a graph on $n$ vertices. Packing (or tiling) problems in extremal graph theory are investigations of conditions under which $G$ must contain many vertex disjoint copies of $H$ (as subgraphs), where minimum degree conditions are studied the most. An $H$-matching of $G$ is a subgraph of $G$ which consists of vertex-disjoint copies of $H$. A perfect $H$-matching, or $H$-factor, of $G$ is an $H$-matching consisting of $\lfloor n/h \rfloor$ copies of $H$. Let $K_k$ denote the complete graph on $k$ vertices. The celebrated theorem of Hajnal and Szemerédi [6] says that every $n$-vertex graph $G$ with $\delta(G) \geq (k-1)n/k$ contains a $K_k$-factor (see [11] for another proof).

Using the Regularity Lemma of Szemerédi [25], researchers have generalized this theorem for packing arbitrary $H$ [1, 15, 24, 16]. Results and methods for packing problems can be found in the survey of Kühn and Osthus [17].

In this paper we consider multipartite packing, which restricts $G$ to be a $k$-partite graph for $k \geq 2$. A $k$-partite graph is called balanced if its partition sets have the same size. Given a $k$-partite graph $G$, it is natural to consider $\delta^*(G)$, the minimum degree from a vertex in one partition set to any other partition set. When $k = 2$, $\delta^*(G)$ is simply $\delta(G)$. In addition, given a graph $H$ and a positive integer $t$, the graph $H(t)$ denotes the blow-up of $H$, obtained by replacing vertices $a_i$ with sets $A_i$ of size $t$, and edges $a_i a_j$ with complete bipartite graphs between $A_i$ and $A_j$.

Let $\mathcal{G}_k(n)$ denote the family of balanced $k$-partite graphs with $n$ vertices in each of its partition sets. It is easy to see (e.g. using the König-Hall Theorem) that every bipartite graph $G \in \mathcal{G}_2(n)$ with $\delta^*(G) \geq n/2$ contains a 1-factor. Fischer [5] conjectured that if $G \in \mathcal{G}_k(n)$ satisfies

$$\delta^*(G) \geq \frac{k-1}{k} n,$$

then $G$ contains a $K_k$-factor and proved the existence of an almost $K_k$-factor for $k = 3, 4$. Magyar and Martin [20] noticed that the condition (1) is not sufficient for odd $k$. Given positive integers $k$, let $\Gamma_k$ denote the graph with vertices $a_{ij}$, $i, j = 1, \ldots, k$ and edges defined in two cases. First, $a_{ij} \sim a_{i'j'}$ if $i \neq i'$, $j \neq j'$ and either $j$ or $j'$ is in $[k - 2]$. Second, $a_{ij} \sim a_{i'j}$ if $i \neq i'$ and $j = k - 1, k$. It is easy to check that the blow-up $\Gamma_k(t)$ satisfies (1) but when $k, t$ are odd, it does not contain a $K_k$-factor.

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Theorem 1 ([20]). There exists an integer $n_0$ such that the following holds for all $n \geq n_0$. Let $G \in G_3(n)$ be a balanced tripartite graph with $\delta^*(G) \geq 2n/3$, then $G$ contains a $K_3$-factor unless $n$ is an odd multiple of 3 and $G \cong \Gamma_3(n/3)$.

On the other hand, Martin and Szemerédi [21] proved that (1) for $k = 4$.

Theorem 2 ([21]). There exists an integer $n_0$ such that if $n \geq n_0$ and $G \in G_k(n)$ satisfies $\delta^*(G) \geq 3n/4$, then $G$ contains a $K_k$-factor.

Recently Keevash and Mycroft [9] and independently Lo and Markström [19] proved that Fischer’s conjecture is asymptotically true, namely, $\delta^*(G) \geq \frac{k-1}{k} n + o(n)$ guarantees a $K_k$-factor for all $k \geq 3$. Very recently, Keevash and Mycroft [10] proved the exact result under (1) for $k \geq 3$.

In this paper we give a new proof of Theorems 1 and 2 by the absorbing method. Our approach is similar to that of [19] (in contrast, a geometric approach was employed in [9] and [10]). However, in order to prove exact results by the absorbing lemma, one can only assume $\delta$ is similar to that of [19] (in contrast, a geometric approach was employed in [9] and [10]). However, this work allows us to improve the almost $\Delta$-extremal.

The absorbing method, initiated by Rödl, Rucinski, and Szemerédi [23], has been shown to be effective handling extremal problems in graphs and hypergraphs. One example is the reproof of Posa’s conjecture by Levitt, Sárközy, and Szemerédi [18], while the original proof of Komlós, Sárközy, and Szemerédi [13] used the Regularity Lemma. Our paper is another example of replacing the regularity method with the absorbing method. Compared with the threshold $n_0$ in Theorems 1 and 2 derived from the Regularity Lemma, the value of our $n_0$ is much smaller.

Before presenting our proof, let us first recall the approach used in [20, 21]. Given a $k$-partite graph $G \in G_k(n)$ with parts $V_1, \ldots, V_k$, the authors said that $G$ is $\Delta$-extremal if each $V_i$ contains a subset $A_i$ of size $[n/k]$ such that the density $d(A_i, A_j) \leq \Delta$ for all $i \neq j$. Using standard but involved graph theoretic arguments, they solved the extremal case for $k = 3, 4$ [20, Theorem 3.1], [21, Theorem 2.1].

Theorem 3. Let $k = 3, 4$. There exists $\Delta$ and $n_0$ such that the following holds. Let $n \geq n_0$ and $G \in G_k(n)$ be a $k$-partite graph satisfying (1). If $G$ is $\Delta$-extremal, then $G$ contains a $K_k$-factor unless $n$ is an odd multiple of 3 and $G \cong \Gamma_3(n/3)$.

To handle the non-extremal case, they proved the following lemma ([20, Lemma 2.2] and [21, Lemma 2.2]).

Lemma 4 (Almost Covering Lemma). Let $k = 3, 4$. Given $\Delta > 0$, there exists $\alpha > 0$ such that for every graph $G \in G_k(n)$ with $\delta^*(G) \geq (1 - 1/k)n - \alpha$ either $G$ contains an almost $K_k$-factor that leaves at most $C$ vertices uncovered or $G$ is $\Delta$-extremal.

To improve the almost $K_k$-factor obtained from Lemma 4, they used the Regularity Lemma and Blow-up Lemma [14]. Here is where we need our absorbing lemma whose proof is given in Section 2. Our lemma actually gives a more detailed structure than what is needed for the extremal case when $G$ does not satisfy the absorbing property.

We need some definitions. Given positive integers $k$ and $r$, let $\Theta_{k \times r}$ denote the graph with vertices $a_{ij}, i = 1, \ldots, k, j = 1, \ldots, r$, and $a_{ij}$ is adjacent to $a_{i'j'}$ if and only if $i \neq i'$ and $j \neq j'$. Recall that, given a positive integer $t$, the graph $\Theta_{k \times r}(t)$ denotes the blow-up of $\Theta_{k \times r}$. Given $\epsilon, \Delta > 0$ and $t \geq 1$ (not necessarily an integer), we say that a $k$-partite graph $G$ is $(\epsilon, \Delta)$-approximate to $\Theta_{k \times r}(t)$ if each of its partition set $V_i$ can be partitioned into $\bigcup_{i=1}^t V_{ij}$ such that $|V_{ij}| - t | \leq \epsilon t$ for all $i, j$ and $d(V_{ij}, V_{ij}) \leq \Delta$ whenever $i \neq i'$ and $j \neq j'$.

Lemma 5 (Absorbing Lemma). Let $k \geq 3$. Given $\Delta > 0$, there exists $\alpha > 0$ and an integer $n_1 > 0$ such that the following holds. Let $n \geq n_1$ and $G \in G_k(n)$ be a $k$-partite graph on $V_1 \cup \cdots \cup V_k$ such that $\delta^*(G) \geq (1 - 1/k)n - \alpha$. Then either of the following cases holds.
(1) $G$ contains a $K_k$-matching $M$ of size $|M| \leq 2(k-1)\alpha^{4k-2}n$ in $G$ such that for every $W \subseteq V \setminus V(M)$ with $|W \cap V_1| = \cdots = |W \cap V_k| \leq \alpha^{8k-6}n/4$, there exists a $K_k$-matching covering exactly the vertices in $V(M) \cup W$.

(2) We may remove some edges from $G$ so that the resulting graph satisfies the minimum degree condition and is $(\Delta/6, \Delta/2)$-approximately $\Theta_{k \times k}(\frac{n}{k})$.

The $K_k$-matching $M$ in Lemma 5 has the so-called absorbing property: it can absorb any balanced set with a much smaller size.

Proof of Theorems 1 and 2. Let $k = 3, 4$. Let $\alpha \ll \Delta$, where $\Delta$ is given by Theorem 3 and $\alpha$ satisfies both Lemmas 4 and 5. Suppose that $n$ is sufficiently large. Let $G \in \mathcal{G}_k(n)$ be a $k$-partite graph satisfying (1). By Lemma 5, either $G$ contains a subgraph which is $(\Delta/2, \Delta/6)$-approximate to $\Theta_{k \times k}(\frac{n}{k})$ or $G$ contains an absorbing $K_k$-matching $M$. In the former case, for $i = 1, \ldots, k$, we add or remove at most $\frac{\Delta n}{6k}$ vertices from $V_i$ to obtain a set $A_i \subset V_i$ of size $|n/k|$. For $i \neq i'$, we have

$$e(A_i, A_{i'}) \leq e(V_i, V_{i'1}) + \frac{\Delta n}{6k} (|A_i| + |A_{i'}|)$$

$$\leq \frac{\Delta}{2} |V_{i1}||V_{i'1}| + 2\frac{\Delta n}{6k} \left\lfloor \frac{n}{k} \right\rfloor$$

$$\leq \Delta \left(1 + \frac{\Delta}{6}\right)^2 \frac{n}{k} + \frac{\Delta n}{3k} \left\lfloor \frac{n}{k} \right\rfloor$$

$$\leq \Delta \left\lfloor \frac{n}{k} \right\rfloor \left\lfloor \frac{n}{k} \right\rfloor,$$

which implies that $d(A_i, A_{i'}) \leq \Delta$. Thus $G$ is $\Delta$-extremal. By Theorem 3, $G$ contains a $K_k$-factor unless $n$ is an odd multiple of 3 and $G \cong \Gamma_3(n/3)$. In the latter case, $G$ contains a $K_k$-matching $M$ of size $|M| \leq 2(k-1)\alpha^{4k-2}n$ such that for every $W \subseteq V \setminus V(M)$ with $|W \cap V_1| = \cdots = |W \cap V_k| \leq \alpha^{8k-6}n/4$, there exists a $K_k$-matching on $V(M) \cup W$. Let $G' = G \setminus V(M)$ be the induced subgraph of $G$ on $V(G) \setminus V(M)$, and $n' = |V(G')|$. Clearly $G'$ is balanced. As $\alpha \ll 1$, we have

$$\delta'(G') \geq \delta'(G) - |M| \geq \left(1 - \frac{1}{k}\right)n - 2(k-1)\alpha^{4k-2}n \geq \left(1 - \frac{1}{k} - \alpha\right)n'. $$

By Lemma 4, $G'$ contains a $K_k$-matching $M'$ such that $|V(G') \setminus V(M')| \leq C$. Let $W = V(G') \setminus V(M')$. Clearly $|W \cap V_i| = \cdots = |W \cap V_k|$. Since $C/k \leq \alpha^{8k-6}n/4$, by the absorbing property of $M$, there is a $K_k$-matching $M''$ on $V(M) \cup W$. This gives the desired $K_k$-factor $M' \cup M''$ of $G$. □

Remarks.

- Since our Lemma 5 works for all $k \geq 3$, it has the potential of proving a general multipartite Hajnal-Szemerédi theorem by the absorbing method. To do it, one only needs to prove Theorem 4 and Lemma 4 for $k \geq 5$.

- Since our Lemma 5 gives a detailed structure of $G$ when $G$ does not have desired absorbing $K_k$-matching, it has the potential of simplifying the proof of the extremal case. Indeed, if one can refine Lemma 4 such that it concludes that $G$ either contains an almost $K_k$-factor or it is approximate to $\Theta_{k \times k}(\frac{n}{k})$ or $\Gamma_{k}(\frac{n}{k})$, then in Theorem 3 we may assume that $G$ is actually approximate to these extreme graphs.

- Using the Regularity Lemma, researchers have obtained results on packing arbitrary graphs in $k$-partite graphs, see [26, 8, 3, 2] for $k = 2$ and [22] for $k = 3$. With the help of the regular results of Keevash–Mycroft [9, 10] and Lo-Markström [19], it seems not very difficult to extend these results to the $k \geq 4$ case (though exact results may be much harder). However, it seems difficult to replace the regularity method by the absorbing method for these problems.
2. Proof of the Absorbing Lemma

In this section we prove the Absorbing Lemma (Lemma 5). We first introduce the concepts of reachability.

**Definition 6.** In a graph $G$, a vertex $x$ is reachable from another vertex $y$ by a set $S$ if there exists an $S \subseteq V(G)$ satisfying that both $G[x \cup S]$ and $G[y \cup S]$ contain $K_k$-factors. In this case, we say $S$ connects $x$ and $y$.

The following lemma plays a key role in constructing absorbing structures. We postpone its proof to the end of the section.

**Lemma 7** (Reachability Lemma). Let $k, \Delta, \alpha$, and $G$ be given as in Lemma 5. Then either of the following cases holds.

1. For any $x$ and $y$ in $V_i$, $i \in [k]$, $x$ is reachable from $y$ by either at least $\alpha^3 n^{k-1} (k-1)$-sets or at least $\alpha^3 n^{2k-1} (2k-1)$-sets in $G$.
2. We may remove some edges from $G$ so that the resulting graph satisfies the minimum degree condition and is $(\Delta/6, \Delta/2)$-approximately $\Theta_{k \times k}(\frac{n}{k})$.

With the aid of Lemma 7, the proof of Lemma 5 becomes standard counting and probabilistic arguments, as shown in [7, Lemma 2.4].

**Proof of Lemma 5.** We assume that $G$ does not satisfy the second property stated in the lemma.

For a crossing $k$-tuple $T = (v_1, \ldots, v_k)$, with $v_i \in V_i$ for $i = 1, \ldots, k$, we call a $2k(2k-1)$-element set $A$ an absorbing $2k(k-1)$-set for $T$ if both $G[A]$ and $G[A \cup T]$ contain $K_k$-factors. Let $\mathcal{L}(T)$ denote the family of all $2k(k-1)$-sets that absorb $T$.

**Claim 8.** For every crossing $k$-tuple $T$, we have $|\mathcal{L}(T)| > \alpha^{4k-3} n^{2k(k-1)}$.

**Proof.** Fix a crossing $k$-tuple $T$. First we try to find a copy of $K_k$ containing $v_1$ and avoiding $v_2, \ldots, v_k$. By the minimum degree condition, there are at least

$$\prod_{i=2}^k \left(n - 1 - (i-1) \left(\frac{1}{k} + \alpha\right)n\right)$$

such copies of $K_k$. When $n \geq 3k^2$ and $\frac{1}{\alpha} \geq 3k^2$, we have $(k-1)\alpha n + 1 \leq n/(3k)$ and thus the number above is at least

$$\prod_{i=2}^k \frac{k - (i-1) - \frac{1}{\alpha}}{k} n \geq \left(\frac{n}{k}\right)^{k-1}.$$

Fix such a copy of $K_k$ on $\{v_1, u_2, u_3, \ldots, u_k\}$. Consider $u_2$ and $v_2$. By Lemma 7 and the assumption that $G$ does not satisfy the second property of the lemma, we can find at least $\alpha^3 n^{k-1}$ $(k-1)$-sets or $\alpha^3 n^{2k-1} (2k-1)$-sets to connect $u_2$ and $v_2$. If $S$ is a $(k-1)$-set that connects $u_2$ and $v_2$, then $S \cup K$ also connects $u_2$ and $v_2$ for any $k$-set $K$ such that $G[K] \cong K_k$ and $K \cap S = \emptyset$.

There are at least

$$(n-2) \prod_{i=2}^k \left(n - 1 - (i-1) \left(\frac{1}{k} + \alpha\right)n\right) \geq \frac{n}{2} \left(\frac{n}{k}\right)^{k-1}$$

copies of $K_k$ in $G$ avoiding $u_2, v_2$ and $S$. If there are at least $\alpha^3 n^{k-1} (k-1)$-sets that connect $u_2$ and $v_2$, then at least

$$\alpha^3 n^{k-1} \frac{n}{2} \left(\frac{n}{k}\right)^{k-1} \left(\frac{1}{2k-1}\right) \geq 2\alpha^4 n^{2k-1}$$

$(2k-1)$-sets connect $u_2$ and $v_2$ because a $(2k-1)$-set can be counted at most $\binom{2k-1}{k-1}$ times. Since $2\alpha^4 < \alpha^3$, we can assume that there are always at least $2\alpha^4 n^{2k-1} (2k-1)$-sets connecting $u_2$ and
v_2$. We inductively choose disjoint $(2k-1)$-sets that connects $v_i$ and $u_i$ for $i = 2, \ldots, k$. For each $i$, we must avoid $T$, $u_2, \ldots, u_k$, and $i-2$ previously selected $(2k-1)$-sets. Hence at least $2\alpha^i n^{2k-1} - (2k-1)(i-1)n^{2k-2} > \alpha^i n^{2k-1}$ $(2k-1)$-sets can be selected for each $i \geq 2$. Putting all these together, we have

$$|\mathcal{L}(T)| \geq \left(\frac{n}{2k}\right)^{k-1} \cdot (\alpha^i n^{2k-1})^{k-1} > \alpha^{4k-3} n^{2k(k-1)}. \tag{1}$$

Every set $S \in \mathcal{L}(T)$ is balanced because $|S \cap V_1| = \cdots = |S \cap V_k| = 2(k-1)$. Note that there are $(\binom{n}{2(k-1)})^k$ balanced $2k(k-1)$-sets in $G$. Let $\mathcal{F}$ be the random family of $2k(k-1)$-sets obtained by selecting each balanced $2k(k-1)$-set from $V(G)$ independently with probability $p := \alpha^{4k-3} n^{1-2k(k-1)}$. Then by Chernoff’s bound, since $n$ is sufficiently large, with probability $1 - o(1)$, the family $\mathcal{F}$ satisfies the following properties:

$$|\mathcal{F}| \leq 2E(|\mathcal{F}|) \leq 2p \left(\frac{n}{2(k-1)}\right)^k \leq \alpha^{4k-2} n, \tag{2}$$

$$|\mathcal{L}(T) \cap \mathcal{F}| \geq \frac{1}{2} E(|\mathcal{L}(T) \cap \mathcal{F}|) \geq \frac{1}{2} p|\mathcal{L}(T)| \geq \frac{\alpha^{8k-6} n}{2} \quad \text{for every crossing $k$-tuple } T. \tag{3}$$

Let $Y$ be the number of intersecting pairs of members of $\mathcal{F}$. Since each fixed balanced $2k(k-1)$-set intersects at most $2k(k-1)\binom{n-1}{2(k-1)-1}\binom{n}{2(k-1)}^{k-1}$ other balanced $2k(k-1)$-sets in $G$,

$$E(Y) \leq p^2 \left(\frac{n}{2(k-1)}\right)^k 2k(k-1)\binom{n-1}{2(k-1)-1}\binom{n}{2(k-1)}^{k-1} \leq \frac{1}{8} \alpha^{8k-6} n.$$ 

By Markov’s bound, with probability at least $\frac{1}{2}$, $Y \leq \alpha^{8k-6} n/4$. Therefore, we can find a family $\mathcal{F}$ satisfying (2), (3) and having at most $\alpha^{8k-6} n/4$ intersecting pairs. Remove one set from each of the intersecting pairs and all non-absorbing sets from $\mathcal{F}$, we get a subfamily $\mathcal{F}'$ consisting of pairwise disjoint absorbing $2k(k-1)$-sets which satisfies $|\mathcal{F}'| \leq |\mathcal{F}| \leq \alpha^{4k-2} n$ and for all crossing $T$,

$$|\mathcal{L}(T) \cap \mathcal{F}'| \geq \frac{\alpha^{8k-6} n}{2} - \frac{\alpha^{8k-6} n}{4} \geq \frac{\alpha^{8k-6} n}{4}. \tag{4}$$

Since $\mathcal{F}'$ consists of disjoint absorbing sets and each absorbing set is covered by a $K_k$-matching, $V(\mathcal{F}')$ is covered by some $K_k$-matching $M$. Since $|\mathcal{F}'| \leq \alpha^{4k-2} n$, we have $|M| \leq 2k(k-1)\alpha^{4k-2} n/k = 2(k-1)\alpha^{4k-2} n$. Now consider a balanced set $W \subseteq V(G) \setminus V(\mathcal{F}')$ such that $|W \cap V_i| = \cdots = |W \cap V_k| \leq \alpha^{8k-6} n/4$. Arbitrarily partition $W$ into at most $\alpha^{8k-6} n/4$ crossing $k$-tuples. We absorb each of the $k$-tuples with a different $2k(k-1)$-set from $\mathcal{L}(T) \cap \mathcal{F}'$. As a result, $V(\mathcal{F}') \cup W$ is covered by a $K_k$-matching, as desired. \hfill \Box

The rest of the paper is devoted to prove Lemma 7. First we prove a useful lemma. A weaker version of it appears in [21, Proposition 1.4] with a brief proof sketch. Our proof contains all details as it has a similar structure as the one of Lemma 5.

**Lemma 9.** Let $k \geq 2$ be an integer, $t \geq 1$ and $\epsilon \ll 1$. Let $H$ be a $k$-partite graph on $V_1 \cup \cdots \cup V_k$ such that $|V_i| \geq (k-1)(1-\epsilon)t$ for all $i$ and each vertex is nonadjacent to at most $(1+\epsilon)t$ vertices in each of the other color classes. Then either $H$ contains at least $\epsilon^2 t^k/2$ copies of $K_k$, or $H$ is $(16k^4 \epsilon^{1/2k-2}, 16k^4 \epsilon^{1/2k-2})$-approximately $\Theta_{k \times (k-1)}(t)$.

**Proof.** First we derive an upper bound for $|V_i|$, $i \in [k]$. Suppose for example, that $|V_k| \geq (k-1)(1+ \epsilon)t + ct$. Then if we greedily construct copies of $K_k$ while choosing the last vertex from $V_k$, by the
minimum degree condition and $\epsilon \ll 1$, we must have at least
\[
|V_1| \cdot (|V_2| - (1 + \epsilon)t) \cdots (|V_{k-1}| - (k - 2)(1 + \epsilon)t) \cdot (|V_k| - (k - 1)(1 + \epsilon)t) \\
\geq (k-1)(1-\epsilon)t \cdot (k-2-\epsilon)t \cdots (1-(2k-3)\epsilon)t \cdot ct \\
\geq \left(k-1 - \frac{1}{k}\right)(k-2 - \frac{1}{k}) \cdots (1-\frac{1}{k})ct^k \geq \epsilon^2tk^k
\]
copies of $K_k$ in $H$, and we are done. We thus assume that for all $i$,
\[
|V_i| \leq (k-1)(1+\epsilon)t + ct < (k-1)(1+2\epsilon)t.
\]
Now we proceed by induction on $k$. The base case is $k = 2$. If $H$ has at least $\epsilon^2t^2$ edges, then we are done. Otherwise $e(H) < \epsilon^2t^2$. Using the lower bound for $|V_1|$, we obtain that
\[
d(V_1, V_2) < \frac{\epsilon^2t^2}{|V_1| \cdot |V_2|} \leq \frac{\epsilon^2}{(1-\epsilon)^2} < \epsilon.
\]
Hence $H$ is $(2\epsilon, \epsilon)$-approximately $\Theta_{2 \times 1}(t)$. When $k = 2$, $16\epsilon^4t^{1/2k-2} = 256\epsilon$, so we are done.

Now assume that $k \geq 3$ and the conclusion holds for $k - 1$. Let $H$ be a $k$-partite graph satisfying the assumptions and assume that $H$ contains less than $\epsilon^2tk^2/2$ copies of $K_k$.

For simplicity, write $N_i(v) = N(v) \cap V_i$ for any vertex $v$. Define
\[
V'_1 = \{v \in V_1 : |N_i(v)| \geq (k-2)(1+\epsilon)t + ct \text{ for some } i > 1\}.
\]
We now show that $|V'_1|$ must be small. Suppose that $v \in V'_1$, and without loss of generality, assume $|N_2(v)| \geq (k-2)(1+\epsilon)t + ct$. We greedily construct copies of $K_{k-1}$ in $N(v)$ while choosing the last vertex from $N_2(v)$. By the minimum degree condition and $\epsilon \ll 1$, we can find at least
\[
(|V_3| - (1+\epsilon)t) \cdot (|V_4| - 2(1+\epsilon)t) \cdots (|V_{k-1}| - (k-2)(1+\epsilon)t) \cdot ct \\
\geq (k-2-k\epsilon)t \cdot (k-3-k\epsilon)t \cdots (1-(2k-3)\epsilon)tct \geq \epsilon^{k-1}
\]
copies of $K_{k-1}$ in $N(v)$. Together with $v$, these give at least $\epsilon t^{k-1}/2$ copies of $K_k$ in $H$. If $|V'_1| \geq ct$, then we obtain at least $\epsilon^2tk^2/2$ copies of $K_k$ in $H$, a contradiction.

We thus assume that $|V'_1| < ct$. Let $\tilde{V}_1 = V_1 \setminus V'$. For $v \in \tilde{V}_1$, by the definition of $V'_1$ and the minimum degree condition, we have
\[
(k-1)(1-\epsilon)t - (1+\epsilon)t \leq |N_i(v)| \leq (k-2)(1+\epsilon)t + ct. \tag{4}
\]
Let $H_v$ be the induced subgraph of $H$ on $\bigcup_{i \geq 2} N_i(v)$. Then $H_v$ is a $(k - 1)$-partite graph satisfying all the assumptions of Lemma 9 with parameter $2\epsilon$. By induction hypothesis, either $H_v \approx \Theta_{(k-1) \times (k-2)}(t)$ with parameter
\[
\epsilon' := 16(k-1)^4(2\epsilon)^{1/2k-3}
\]
or $H_v$ contains at least $(2\epsilon)^2tk^{2k-1}/2 = 2\epsilon^2tk^{2k-1}$ copies of $K_{k-1}$.

If $H_v$ contains at least $2\epsilon^2tk^{2k-1}$ copies of $K_{k-1}$ for all $v \in \tilde{V}_1$, then there are at least $(|V_1| - ct)2\epsilon^2tk^{2k-1} > \epsilon^2tk^2/2$ copies of $K_k$ in $H$, a contradiction. Fix a vertex $v_0 \in \tilde{V}_1$ such that $H_{v_0}$ is $(\epsilon', \epsilon')$-approximate to $\Theta_{(k-1) \times (k-2)}(t)$. This means that we can partition $N_i(v_0)$ into $A_{i2} \cup \cdots \cup A_{ik}$ for $i = 2, \ldots, k$ such that
\[
\forall 2 \leq i, j \leq k, (1-\epsilon')t \leq |A_{ij}| \leq (1+\epsilon')t \text{ and } (5)
\]
\[
\forall 2 \leq i, i', j \leq k \text{ with } i \neq i', d(A_{ij}, A_{i'j}) \leq \epsilon'. \tag{6}
\]
Let $A_{11} = V_1 \setminus N_i(v_0)$. By assumptions and (4), we have
\[
(1-(2k-2)\epsilon)t = (k-1)(1-\epsilon)t - ((k-2)(1+\epsilon)t + ct) \leq |A_{11}| < (1+\epsilon)t. \tag{7}
\]
Let $A_{ij}^c = V_i \setminus A_{ij}$ denote the complement of $A_{ij}$. Let $\bar{e}(A, B) = |A||B| - e(A, B)$ denote the number of non-edges between two disjoint sets $A$ and $B$, and $d(A, B) = \bar{e}(A, B)/(|A||B|) = 1 - d(A, B)$.
Given two disjoint sets $A$ and $B$ (with density close to one) and $\alpha > 0$, we call a vertex $a \in A$ is $\alpha$-typical to $B$ if $\deg_B(a) \geq (1 - \alpha)|B|$

**Claim 10.** Let $2 \leq i \neq i' \leq k$, $1 \leq j \neq j' \leq k - 1$.

1. $d(A_{ij}, A_{ij'}) \geq 1 - 3\epsilon'$ and $d(A_{ij}, A_{ij''}) \geq 1 - 3\epsilon'$.
2. All but at most $\sqrt{3\epsilon'}$ vertices in $A_{ij}$ are $\sqrt{3\epsilon'}$-typical to $A_{ij'}$; at most $\sqrt{3\epsilon'}$ vertices in $A_{ij}$ are $\sqrt{3\epsilon'}$-typical to $A_{ij''}$.

**Proof.** (1). Since $A_{ij} \cap \bigcup_{j' \neq j} A_{ij'}$, the second assertion $d(A_{ij}, A_{ij'}) \geq 1 - 3\epsilon'$ immediately follows from the first assertion $d(A_{ij}, A_{ij'}) \geq 1 - 3\epsilon'$. Thus it suffices to show that $d(A_{ij}, A_{ij'}) \geq 1 - 3\epsilon'$, equivalently $d(A_{ij}, A_{ij'}) \leq 3\epsilon'$.

Assume $j \geq 2$. By (6), we have $e(A_{ij}, A_{ij'}) \leq \epsilon'|A_{ij}|[A_{ij'}]$. Then $\hat{e}(A_{ij}, A_{ij'}) \geq (1 - \epsilon'|A_{ij}|[A_{ij'}]).$

By the minimum degree condition and (5),

$$\hat{d}(A_{ij}, A_{ij'}) \leq (1 + \epsilon)t - (1 - \epsilon'|A_{ij'}|)|A_{ij}|
\leq (1 + \epsilon)\frac{t}{|A_{ij'}|} \leq (1 + \epsilon)t \leq 3\epsilon',$$

where the last inequality holds because $\epsilon \ll \epsilon' \ll 1$.

(2) Given two disjoint sets $A$ and $B$, if $\hat{d}(A, B) \leq \alpha$ for some $\alpha > 0$, then at most $\sqrt{\alpha}|A|$ vertices $a \in A$ satisfy $\deg_B(a) < (1 - \sqrt{\alpha})|B|$. Hence Part (2) immediately follows from Part (1). \hfill \Box

Now we want to study the structure of $\tilde{V}_1$. Let $\epsilon'' = 2k\sqrt{\epsilon'}$. Define

$$A_{10} = \{v \in \tilde{V}_1 : 3 \leq i, i' \leq k \text{ and } j \in [k - 1] \text{ such that } \deg_{A_{ij}}(v) \geq \epsilon''|A_{ij}|	ext{ and }\deg_{A_{ij'}}(v) < \epsilon''|A_{ij'}|\}.$$

Note that if $v \in V_1$ satisfies $\deg_{A_{ij}}(v) < \epsilon''|A_{ij}|$, then $N(v)$ misses at least $(1 - \epsilon'')|A_{ij}|$ vertices in $A_{ij}$. By the minimum degree condition, (5) and (7),

$$|A_{ij'} \setminus N(v)| \leq (1 + \epsilon)t - (1 - \epsilon'')|A_{ij}| \leq (1 + \epsilon)t - (1 - \epsilon'')t < (\epsilon + \epsilon' + \epsilon'').$$

Let $N(v_1 \cdots v_m) = \bigcap_{1 \leq i \leq m} N(v_i)$. 

**Claim 11.** $|A_{10}| \leq ct$.

**Proof.** Let $v_1 \in A_{10}$. Without loss of generality, assume $\deg_{A_{21}}(v) \geq \epsilon''|A_{21}|$ and $\deg_{A_{k1}}(v) < \epsilon''|A_{k1}|$. (since we will use $|A_{ij}| \geq (1 - \epsilon')t$ for $i \geq 2$ and $j \geq 1$, we do not need to distinguish the $j = 1$ case from the $j \geq 2$ case).

We greedily construct the subgraphs of $K_{k - 1}$ on $N(v_1)$. First choose a vertex $v_2 \in N_{A_{21}}(v_1)$ such that $v_2$ is $\sqrt{3\epsilon'}$-typical to $A_{21}'$. By Claim 10, there are at least $(\epsilon'' - \sqrt{3\epsilon'})|A_{21}| \geq 4\sqrt{\epsilon'}t$ such vertices (using the fact $\epsilon'' \geq 6\sqrt{\epsilon'}$). Then we simply choose a vertex $v_3 \in N_{A_3}(v_1v_2)$. By the minimum degree condition, at least $|V_3| - 2(1 + \epsilon)t$ vertices can be $v_3$. We choose $v_4, \ldots, v_{k-1}$ greedily, for example, at least $|V_{k-1}| - (k - 2)(1 + \epsilon)t$ vertices can be selected as $v_{k-1}$. Finally we choose $v_k \in N_{A_{k1}}(v_1 \cdots v_{k-1})$. Using the minimum degree condition for $v_3, \ldots, v_{k-1}$, the bound (8) for $v_1$, and the fact that $v_2$ is $\sqrt{3\epsilon'}$-typical to $A_{11}'$, there are at least

$$(1 - \sqrt{3\epsilon'})|A_{k1}'| - (\epsilon + \epsilon' + \epsilon'')t - (k - 3)(1 + \epsilon)t \geq \frac{t}{2}.$$
vertices can be \( v_k \). Consequently there are at least
\[
4\sqrt{\epsilon'} \cdot (|V_3| - 2(1 + \epsilon)t) \cdots (|V_{k-1}| - (k - 2)(1 + \epsilon)t) \cdot \frac{t}{2} \\
\geq 4\sqrt{\epsilon'}t \cdot \left( k - 3 - \frac{1}{2} \right) \cdot \frac{t}{2} \cdots \frac{t}{2} \\
\geq \sqrt{\epsilon'}t^{k-1}
\]
copies of \( K_{k-1} \) in \( N(v_1) \). Since each vertex in \( A_{10} \) gives at least \( \sqrt{\epsilon'}t^{k-1} \) copies of \( K_k \), if \( |A_{10}| \geq t \), then we get at least \( \epsilon^2 t^k \) copies of \( K_k \), a contradiction. \( \square \)

For each vertex \( v \) in \( \hat{V}_1 \setminus A_{10} \) and each \( j \in [k - 1] \), either \( \deg_{A_{10}}(v) < \epsilon''|A_{10}| \) for \( 2 \leq i \leq k \) or \( \deg_{A_{10}}(v) \geq \epsilon''|A_{10}| \), for \( 2 \leq i \leq k \). By the minimum degree condition, there exists at most one \( j \in [k - 1] \) such that \( \deg_{A_{10}}(v) < \epsilon''|A_{10}| \). For \( j \in [k - 1] \), we let \( A_{1j} \) be the set of vertices \( v \) in \( \hat{V}_1 \) such that \( \deg_{A_{1j}}(v) < \epsilon''|A_{1j}| \) (actually we can use any \( A_{1j} \) with \( i \geq 2 \)). Let \( A'_{10} = \hat{V}_1 \setminus \bigcup_{j=0}^{k-1} A_{1j} \).

**Claim 12.** \( |A'_{10}| \leq t \).

**Proof.** Fix \( v_1 \in A'_{10} \). As in the proof of Claim 11, we construct copies of \( K_{k-1} \) in \( N(v_1) \). First select \( k - 1 \) sets \( A_{1j} \) with \( 2 \leq i \leq k \) and \( 1 \leq j \leq k - 1 \) such that no two of them are on the same row or column, for example, \( A_{21}, A_{32}, \ldots, A_{k(k-1)} \). (Note that there are \((k - 1)!\) such choices.) By Claim 10, the density between any two of \( A_{21}, \ldots, A_{k(k-1)} \) is at least \( 1 - 3\epsilon' \). For \( 2 \leq i \leq k \), we select a vertex \( v_i \in N_{A_{1(i-1)}}(v_1 \cdots v_{i-1}) \) such that \( v_i \) is \( \sqrt{3\epsilon'} \)-typical to all \( A_{1(j-1)}, i < j \leq k \). By the definitions of \( A'_{10}, \epsilon'' \) and Claim 10, at least \((\epsilon'' - (k - 2)\sqrt{3\epsilon'})|A_{1(i-1)}| \geq 2\sqrt{\epsilon'}t \) can be selected as \( v_i \). Using the definition of \( \epsilon' \), this gives at least
\[
(k - 1)! \left( 2\sqrt{\epsilon'} \right)^{k-1} = (k - 1)! \left( 2t \right)^{k-1} \left( 16(k - 1)^4(2\epsilon)^{1/2k-3} \right)^{k-1} > ct^{k-1}
\]
copies of \( K_k \). If \( |A'_{10}| \geq t \), then we get at least \( \epsilon^2 t^k \) copies of \( K_k \), a contradiction. \( \square \)

By the definition of \( A_{1j} \), we have
\[
d(A_{1j}, A_{1j}) < \epsilon'' \quad \text{for} \quad i \geq 2, j \geq 1.
\]
This implies that \( |A_{1j}| \leq (1 + \epsilon)t + \epsilon''|A_{1j}| \). For instead, by the minimum degree condition, we have \( \deg_{A_{1j}}(v) > \epsilon''|A_{1j}| \) for all \( v \in A_{1j} \), and consequently \( d(A_{1j}, A_{1j}) > \epsilon'' \), contradicting (9). We thus conclude that
\[
|A_{1j}| \leq \frac{1 + \epsilon}{1 - \epsilon''} \leq (1 + 2\epsilon'')t.
\]
Since \( |V'_1|, |A_{10}|, |A'_{10}| \leq ct \), we have
\[
\left| \bigcup_{j=1}^{k-1} A_{1j} \right| = |V'_1| \setminus \left( V'_1 \cup A_{10} \cup A'_{10} \right) \geq |V_1| - 3ct.
\]
Using (10), we now obtain a lower bound for \( |A_{1j}|, j \in [k - 1] \).
\[
|A_{1j}| \geq (k - 1)(1 - \epsilon)t - (k - 2)(1 + \epsilon'')t - 3ct \geq (1 - 2k\epsilon'')t.
\]
What remains to show is that for \( 2 \leq i, i' \leq k \), \( d(A_{1i}, A_{1i'}) \) is small.

**Claim 13.** \( d(A_{1i}, A_{1i'}) \leq 6\epsilon'' \) for \( 2 \leq i, i' \leq k \).
Proof. Suppose to the contrary, say \( d(A_{(k-1)}, A_{k}) > 6\epsilon'' \). First select \( k-2 \) \( A_{ij} \) with \( 1 \leq i \leq k-2 \) and \( 2 \leq j \leq k-1 \) such that no two of them are on the same row or column, for example, \( A_{12}, A_{23}, \ldots, A_{(k-2)(k-1)} \). We construct copies of \( K_{k-2} \) in these sets as follows. Pick arbitrary \( v_1 \in A_{12} \). For \( 2 \leq i \leq k-2 \), we select \( v_i \in N_{A_{(i+1)}}(v_1 \cdots v_{i-1}) \) such that \( v_i \) is \( 3\epsilon'' \)-typical to \( A_{(k-1)}, A_{k} \) and all \( A_{(j+1)}, i < j \leq k-2 \). By Claim 10 and (8), at least \((1 - (k-2)\sqrt{3\epsilon''})|A_{(i+1)}| - (\epsilon + \epsilon'' + \epsilon')t \geq t/2 \) vertices can be selected as \( v_i \). After selecting \( v_1, \ldots, v_{k-2} \), we select two adjacent vertices \( v_{k-1} \in A_{(k-1)} \) and \( v_k \in A_{k} \) such that \( v_{k-1} \) and \( v_k \) are in \( N(v_1 \cdots v_{k-2}) \). For \( j = k-1, k \), \( N(v_j) \) misses at most \((\epsilon + \epsilon'' + \epsilon')t \) vertices in \( A_j \) and at most \((k-3)\sqrt{3\epsilon''}|A_j| \) vertices of \( A_j \) are not in \( N(v_2 \cdots v_{k-2}) \). Since \( d(A_{(k-1)}, A_{k}) > 6\epsilon'' \) and \( \epsilon'' = 2k\sqrt{\epsilon} \), at least \((k-2)\frac{1}{2}2^{k-2}6\epsilon''t^2 > \epsilon k \) copies of \( K_k \), a contradiction.

In summary, by (5), (7), (10) and (11), we have \((1 - 2k\epsilon'')t \leq |A_{ij}| \leq (1 + 2\epsilon'')t \) for all \( i \) and \( j \). In order to make \( \bigcup_{j=1}^{k-1} A_{ij} \) a partition of \( V_1 \), we move all the vertices of \( V_1 \), \( A_{10} \) to \( A_{11} \). We still have \(|A_{ij}| - t \leq 2k\epsilon''t \) after moving these vertices. On the other hand, by (6), (9), and Claim 13, we have \( d(A_{ij}, A_{ij}) \leq 6\epsilon'' \leq 2k\epsilon'' \) for \( i \neq i' \) and all \( j \) (at present \( d(A_{11}, A_{11}) \leq 2\epsilon'' \) for all \( i \geq 2 \) because \( |A_{11}| \) becomes slightly larger). Therefore \( H \) is \((2k\epsilon'', 2k\epsilon'')\)-approximately \( \Theta_{k \times (k-1)}(t) \). By the definitions of \( \epsilon'' \) and \( \epsilon' \),

\[
2k\epsilon'' = 4k^2\sqrt{\epsilon} = 4k^2\sqrt{16(k-1)^4(2\epsilon)^{1/2k-3}} \leq 16k^4 \epsilon^{1/2k-2},
\]

where the last inequality is equivalent to \((\frac{k-1}{k})^{2} 2^{1/2k-2} \leq 1 \) or \( 2^{1/2k-1} \leq \frac{k-1}{k} \), which holds because

\[
2 \leq 1 + \frac{2k-1}{k-1} \leq (1 + \frac{1}{k-1})^{2k-1} \text{ for } k \geq 2.
\]

This completes the proof of Lemma 9.

We are ready to prove Lemma 7.

Proof of Lemma 7. First assume that \( G \in \mathcal{G}_3(n) \) is minimal, namely, \( G \) satisfies the minimum degree condition but removing any edge of \( G \) will destroy this condition. Note that this assumption is only needed by Claim 19.

Given \( 0 < \Delta \leq 1 \), let

\[
\alpha = \frac{1}{2k} \left( \frac{\Delta}{24k(k-1)\sqrt{2k}} \right)^{2k-1}.
\]

Without loss of generality, assume that \( x, y \in V_1 \) and \( y \) is not reachable by \( \alpha^3n^{k-1} (k-1) \)-sets or \( \alpha^3n^{2k-1} (2k-1) \)-sets from \( x \).

For \( 2 \leq i \leq k \), define

\[
A_{ik} = V_k \cap (N(x) \setminus N(y)), \quad A_{ik} = V_i \cap (N(y) \setminus N(x)), \quad B_i = V_i \cap (N(x) \cap N(y)), \quad A_{i0} = V_i \setminus (N(x) \cup N(y)).
\]

Let \( B = \bigcup_{i \geq 2} B_i \). If there are at least \( \alpha^3n^{k-1} \) copies of \( K_{k-1} \) in \( B \), then \( x \) is reachable from \( y \) by at least \( \alpha^3n^{k-1} (k-1) \)-sets. We thus assume there are less than \( \alpha^3n^{k-1} \) copies of \( K_{k-1} \) in \( B \).

Clearly, for \( i \geq 2 \), \( A_{i1}, A_{ik}, B_i \) and \( A_{i0} \) are pairwise disjoint. The following claim bounds the sizes of \( A_{ik}, B_i \) and \( A_{i0} \).

Claim 14. \( (1 - k^2\alpha)^{\frac{9}{k}} < |A_{i1}|, |A_{ik}| \leq (1 + k\alpha)^{\frac{9}{k}} \).

(2) \((k - 2 - 2\alpha)^2 \leq |B_i| < (k - 2 + k(k - 1)\alpha) \frac{n}{k}\),
(3) \(|A_{i0}| < (k + 1)\alpha n\).

**Proof.** For \(v \in V_i\), and \(i \in [k]\), simply write \(N_i(v) := N(v) \cap V_i\). By the minimum degree condition, we have \(|A_{i1}|, |A_{ik}| \leq (1/k + \alpha)n\). Since \(N_i(x) = A_{i1} \cup B_i\), it follows that
\[
|B_i| \geq \left(\frac{k - 1}{k} - \alpha\right)n - (1/k + \alpha)n. 
\]
(13)

If some \(B_i\), say \(B_k\), has at least \((k - 2 - (k - 1)\alpha)n\) vertices, then we can greedily construct copies of \(K_{k-1}\) in \(B\) while selecting the vertices in \(B_k\) at last. This gives at least \(\prod_{i=2}^{k} |B_i| - (i - 2) \left(\frac{1}{k} + \alpha\right)n\) copies of \(K_{k-1}\) in \(B\). By (13) and \(|B_k| \geq (\frac{k - 2}{k} + (k - 1)\alpha)n\), the number of copies of \(K_{k-1}\) in \(B\) is at least
\[
an \cdot \prod_{i=2}^{k-1} \left(\frac{k - i}{k} - \alpha\right)n - (i - 1) \left(\frac{1}{k} + \alpha\right)n 
\]
\[
= an \cdot \prod_{i=2}^{k-1} \left(\frac{k - i}{k} - i\alpha\right)n 
\]
\[
\geq an \cdot \prod_{i=2}^{k-1} \left(\frac{k - i - 1/2}{k}\right)n \quad \text{because } 2k^2\alpha \leq 1, 
\]
\[
\geq an \cdot \frac{1}{2} \left(n \frac{k}{k}\right)^{k-2} 
\]
\[
\geq \alpha^2 n^{k-1} \quad \text{because } 2k^{k-2}\alpha \leq 1. 
\]

This is a contradiction.

We thus assume that \(|B_i| < (\frac{k - 2}{k} + (k - 1)\alpha)n\) for \(2 \leq i \leq k\), as desired by Part (2). As \(N_i(x) = A_{i1} \cup B_i\), it follows that
\[
|A_{i1}| > (\frac{k - 1}{k} - \alpha)n - (\frac{k - 2}{k} + (k - 1)\alpha)n = (\frac{1}{k} - \alpha)n. 
\]
The same holds for \(|A_{ik}|\) thus Part (1) follows. Finally
\[
|A_{i0}| = |V_i| - |N_i(x)| - |A_{ik}| < n - (\frac{k - 1}{k} - \alpha)n - (\frac{1}{k} - \alpha)n = (k + 1)\alpha n, 
\]
as desired by Part (3). □

Let \(t = n/k\) and \(\epsilon = 2k\alpha\). By the minimum degree condition, every vertex \(u \in B\) is nonadjacent to at most \((1 + k\alpha)n/k < (1 + \epsilon)t\) vertices in other color classes of \(B\). By Claim 14, \(|B_i| \geq (k - 2 - 2k\alpha) \frac{n}{k} = (k - 2 - \epsilon)t \geq (k - 2)(1 - \epsilon)t\). Thus \(G[B]\) is a \((k - 1)\)-partite graph that satisfies the assumptions of Lemma 9. Since we assumed that \(B\) contains less than \(\alpha^2 n^{k-1} < e^{t^2k^{-1}/2} \) copies of \(K_{k-1}\), so \(B\) is \((\alpha', \alpha')\)-approximate to \(\Theta(k-1) \times (k-2)\left(\frac{2}{k}\right)\), where \(\alpha' := 16(k - 1)^4(2k\alpha)^{1/2k^{k-3}}\).

This means that we can partition \(B_i\), \(2 \leq i \leq k\), into \(A_{i2} \cup \cdots A_{i(k-1)}\) such that \((1 - \alpha') \frac{n}{k} \leq |A_{ij}| \leq (1 + \alpha') \frac{n}{k}\) for \(2 \leq j \leq k - 1\) and
\[
\forall 2 \leq i \neq i' \leq k \text{ and } 2 \leq j \leq k - 1, \quad d(A_{ij}, A_{i'j}) \leq \alpha'. 
\]
Together with Claim 14 Part (1), we obtain that (using \(k^2\alpha \leq \alpha'\))
\[
\forall 2 \leq i \leq k \text{ and } 1 \leq j \leq k, \quad (1 - \alpha') \frac{n}{k} \leq |A_{ij}| \leq (1 + \alpha') \frac{n}{k}. 
\]
(15)

Let \(A_{ij}^c = V_i \setminus A_{ij}\) denote the complement of \(A_{ij}\). The following claim is an analog of Claim 10, and its proof is almost the same — after we replace \((1 + \epsilon)t\) with \((1 + k\alpha)n/k\) and \(\epsilon'\) with \(\alpha'\) (and we use \(\alpha \ll \alpha'\)). We thus omit the proof.

**Claim 15.** Let \(2 \leq i \neq i' \leq k, 1 \leq j \neq j' \leq k,\) and \(\{j, j'\} \neq \{1, k\}\).
(1) \( d(A_{ij}, A_{i'j'}) \geq 1 - 3\alpha' \) and \( d(A_{ij}, A_{i'j}) \geq 1 - 3\alpha' \).

(2) All but at most \( \sqrt{3\alpha'} \) vertices in \( A_{ij} \) are \( \sqrt{3\alpha'} \)-typical to \( A_{i'j'} \); at most \( \sqrt{3\alpha'} \) vertices in \( A_{ij} \) are \( \sqrt{3\alpha'} \)-typical to \( A_{i'j'} \). \( \square \)

Now let us study the structure of \( V_1 \). Let \( \alpha'' = 2k\sqrt{\alpha'} \). If \( v \in V_1 \) satisfies deg\(_{A_{ij}}(v) < \alpha''|A_{ij}| \) for some \( i, j \), then \( |A_{ij} \setminus N(v)| \geq (1 - \alpha'')(|A_{ij}| \). By the minimum degree condition and (15),

\[
|A_{ij} \setminus N(v)| \leq (\frac{1}{k} + \alpha)n - (1 - \alpha'')(|A_{ij}| < (k\alpha + \alpha' + \alpha'')\frac{n}{k}. \quad (16)
\]

Let

\[
A_{10} = \{ v \in V_1 : \exists 2 \leq i, i' \leq k \text{ and } j \in [k] \text{ such that deg}_{A_{i,j}}(v) \geq \alpha''|A_{ij}| \\
\text{and deg}_{A_{i,j}}(v) < \alpha''|A_{ij}| \}.
\]

Recall that \( N(xv_1) = N(x) \cap N(v_1) \). Our next claim says that \( |A_{10}| \) must be small because each vertex of \( A_{10} \) provides many reachable \((2k - 1)\)-sets from \( x \) to \( y \). Its proof is similar to the one of Claim 11; the only difference is that for each vertex \( v_1 \in A_{10} \), we repeatedly construct two vertex disjoint copies of \( K_{k-1} \), one in \( N(xv_1) \) and the other one in \( N(yv_1) \) while in Claim 11 we only need to construct copies of \( K_{k-1} \) in \( N(v_1) \).

Claim 16. \( |A_{10}| \leq \alpha n \).

Proof. Let \( v_1 \in A_{10} \). Without loss of generality, assume deg\(_{A_{21}}(v) \geq \alpha''|A_{21}| \) and deg\(_{A_{k1}}(v) < \alpha''|A_{k1}| \).

First we construct a copy of \( K_{k-1} \) on \( \{v_2, \ldots, v_k\} \subset N(xv_1) \). We first choose a vertex \( v_2 \in N_{A_{21}}(v_1) \) such that \( v_2 \) is \( \sqrt{3\alpha'} \)-typical to \( A_{k1} \). By Claim 15, there are at least \( (\alpha'' - \sqrt{3\alpha'})|A_{21}| \geq 4\sqrt{\alpha'}n/k \) such vertices (we use the fact \( \alpha'' = 2k\sqrt{\alpha'} \geq 6\sqrt{\alpha'} \)). Next we choose a vertex \( v_3 \in N_{v_2}(xv_1v_2) \). By the minimum degree condition, at least \( |V_3| - 3(1 + k\alpha)n/k \) vertices can be \( v_3 \). We choose \( v_4, \ldots, v_{k-1} \) greedily, for example, at least \( |V_{k-1}| - (k - 1)(1 + k\alpha)n/k \) vertices can be selected as \( v_{k-1} \). Finally we choose \( v_k \in N_{A_{21}}(xv_1 \ldots v_{k-1}) \). Using the minimum degree condition for \( x, v_3, \ldots, v_{k-1} \), (16) for \( v_1 \), and the fact that \( v_2 \) is \( \sqrt{3\alpha'} \)-typical to \( A_{k1} \), there are at least

\[
(1 - \sqrt{3\alpha'})|A_{k1}| - (k\alpha + \alpha' + \alpha'')\frac{n}{k} - (k - 2)(1 + \alpha)\frac{n}{2k} \geq \frac{n}{2k}
\]

vertices can be chosen as \( v_k \). Totally there are at least

\[
4\sqrt{\alpha'}\frac{n}{k} \cdot (|V_3| - 3(1 + \alpha)n) \cdot |V_{k-1}| - (k - 1)(1 + \alpha)n) \cdot \frac{n}{k} \geq \frac{n}{2k}
\]

\[
\geq 4\sqrt{\alpha'}\frac{n}{k} \cdot (k - 3 - 3k\alpha)\frac{n}{k} \cdot (1 - (k - 1)k\alpha)\frac{n}{k} \cdot \frac{n}{2k}
\]

\[
\geq \sqrt{\alpha'}\left(\frac{n}{k}\right)^{k-1}
\]

copies of \( K_{k-1} \) in \( N(xv_1) \).

Similarly we construct a copy of \( K_{k-1} \) in \( N(yv_1) \) that is disjoint from \( \{v_2, \ldots, v_k\} \). When counting the number of such copies of \( K_{k-1} \), we subtract one from each term in (17) and conclude that at least \( \frac{1}{2}\sqrt{\alpha'}\left(\frac{n}{k}\right)^{k-1} \) copies of \( K_{k-1} \) in \( N(yv_1) \) are disjoint from any fixed \( \{v_2, \ldots, v_k\} \). Therefore, each vertex \( v_1 \in A_{10} \) is contained in at least \( \frac{1}{2}\sqrt{\alpha'}\left(\frac{n}{k}\right)^{2k-2} \) reachable \((2k - 1)\)-sets from \( x \) to \( y \). If \( |A_{10}| \geq \alpha n \), then we get at least \( \alpha n \cdot \frac{1}{2}\sqrt{\alpha'}\left(\frac{n}{k}\right)^{2k-2} \geq \alpha^2 n^{2k-1} \) reachable \((2k - 1)\)-sets from \( x \) to \( y \), a contradiction. \( \square \)

For each vertex \( v \in V_1 \setminus A_{10} \) and each \( j \in [k] \), either deg\(_{A_{ij}}(v) < \alpha''|A_{ij}| \) for \( 2 \leq i \leq k \) or deg\(_{A_{ij}}(v) \geq \alpha''|A_{ij}| \), for \( 2 \leq i \leq k \). By the minimum degree condition, there exists at most one \( j \in [k] \) such that deg\(_{A_{ij}}(v) < \alpha''|A_{ij}| \). For \( j \in [k] \), we let \( A_{ij} \) be the set of vertices \( v \in V_1 \) such that
deg_{A_{ij}}(v) < \alpha''|A_{ij}| \) (actually we can use any \( A_{ij} \) with \( i \geq 2 \)). Let \( A_{ij}' = \mathcal{V}_j \setminus \bigcup_{j=0}^{k} A_{ij} \). Thus for each vertex \( v \in A_{10}' \), we have \( \deg_{A_{ij}'}(v) \geq \alpha''|A_{ij}| \), for all \( i \geq 2 \) and \( j \in [k] \).

As Claim 12, our next claim says that \( |A_{10}'| \) is small. The proof is similar to the one of Claim 11. As in the proof of Claim 16, for each vertex \( v_1 \in A_{10}' \), we construct two vertex disjoint copies of \( K_{k-1} \), one in \( N(x_{v_1}) \) and the other one in \( N(y_{v_1}) \).

**Claim 17.** \( |A_{10}'| \leq \alpha n \).

**Proof.** Fix \( v_1 \in A_{10}' \). First select \( k-1 \) sets \( A_{ij} \) with \( 2 \leq i \leq k \) and \( 1 \leq j \leq k-1 \) such that no two of them are on the same row or column, for example, \( A_{21}, A_{32}, \ldots, A_{k(k-1)} \). (Note that there are \( (k-1)! \) such choices.)

For \( 2 \leq i \leq k \), we select a vertex \( v_i \in N_{A_{i(i-1)}}(v_1 \cdots v_{i-1}) \) such that \( v_i \) is \( \sqrt{3}\alpha' \)-typical to \( A_{j(j-1)} \) for \( j = i+1, \ldots, k \). By Claim 15 and the fact \( \alpha'' = 2k\sqrt{\alpha'} \), at least \( (\alpha'' - (k-2)\sqrt{3}\alpha')|A_{i(i-1)}| \geq 2\sqrt{\alpha'}|2n| \) can be selected as \( v_i \). Using the definition of \( \alpha' \), this procedure gives at least

\[
(k-1)! \left( 2\sqrt{\alpha'} n \right)^{k-1} = (k-1)! \left( \frac{2n}{\alpha} \right)^{k-1} \left( 16(k-1)^2(2\alpha')^{1/2^{k-3}} \right)^{k-1} \geq \alpha n^{k-1}
\]
copies of \( K_{k-1} \) on \( N(x_{v_1}) \). After selecting \( \{v_2, \ldots, v_k\} \), we similarly construct a copy of \( K_{k-1} \) on \( N(y_{v_1}) \) such that it is vertex disjoint from \( \{v_2, \ldots, v_k\} \). Thus at most \( (\alpha'' - (k-2)\sqrt{3}\alpha')\left|A_{i(i-1)}\right| - 1 \geq 2\sqrt{\alpha'}|2n| \) can be selected as \( v_i' \) and there are at least \( \alpha n^{k-1} \) copies of \( K_{k-1} \) in \( N(y_{v_1}) \) that are disjoint from \( \{v_2, \ldots, v_k\} \). Therefore, each vertex \( v_i \in A_{10}' \) is contained in at least \( \alpha^2 n^{2k-2} \) reachable \( (2k-1) \)-sets from \( x \) to \( y \). If \( |A_{10}'| \geq \alpha n \), then we get at least \( \alpha^3 n^{2k-1} \) reachable \( (2k-1) \)-sets from \( x \) to \( y \), a contradiction. \( \square \)

Fix \( j \geq 1 \). By the definition of \( A_{ij} \), we have

\[
d(A_{ij}, A_{ij}) < \alpha'' \text{ for } i \geq 2.
\]

This implies that \( |A_{ij}| \leq \left( \frac{1}{k} + \alpha \right)n + \alpha''|A_{ij}| \). Otherwise, by the minimum degree condition, we have \( \deg_{A_{i,j}}(v) > \alpha''|A_{ij}| \) for all \( v \in A_{ij} \), and consequently \( d(A_{ij}, A_{ij}) > \alpha'' \), contradicting (18). We thus conclude that

\[
|A_{ij}| \leq \frac{1}{1 - \alpha''} n < (1 + 2\alpha'') \frac{n}{k}.
\]

(19)

Since \( |A_{10}'|, |A_{10}'| \leq \alpha n \), we have

\[
\left| \bigcup_{j=1}^{k} A_{ij} \right| = |V_1| \setminus (A_{10} \cup A_{10}') \geq |V_1| - 2\alpha n.
\]

Using (19), we now obtain a lower bound for \( |A_{1j}| \), \( j \in [k] \).

\[
|A_{1j}| \geq n - (k-1)(1 + 2\alpha'') \frac{n}{k} - 2\alpha n \geq (1 - 2k\alpha'') \frac{n}{k}.
\]

(20)

What remains to show is that \( d(A_{1j}, A_{i'j}) \) and \( d(A_{ik}, A_{i'k}) \), \( 2 \leq i, i' \leq k \), are small. First we show that if both densities are reasonably large then we can find enough reachable \( (2k-1) \)-sets from \( x \) to \( y \). The proof resembles the one of Claim 13.

**Claim 18.** For \( 2 \leq i \neq i' \leq k \), either \( d(A_{1i}, A_{1i'}) \leq 6\alpha'' \) or \( d(A_{ik}, A_{i'k}) \leq 6\alpha'' \).

**Proof.** Suppose instead, say \( d(A_{(k-1)j}, A_{(k-1)j'}) > 6\alpha'' \). Fix a vertex \( v_1 \) in \( A_{1j} \), for some \( 2 \leq j \leq k-1 \), say \( v_1 \in A_{12} \). We construct two vertex disjoint copies of \( K_{k-1} \) in \( N(x_{v_1}) \) and \( N(y_{v_1}) \) as follows. To find a copy of \( K_{k-1} \) in \( N(x_{v_1}) \), we select \( k-3 \) sets \( A_{ij} \) with \( 2 \leq i \leq k-2 \) and \( 3 \leq j \leq k-1 \) such that no two of them are on the same row or column, for example, \( A_{23}, \ldots, A_{(k-2)(k-1)} \). For
2 ≤ i ≤ k − 2, we select vi ∈ Nvi,i+1(vi, · · · , vi−1) such that vi is \( \sqrt{3}α'' \)-typical to \( A_{i−1,j} \), \( A_k \) and \( A_{j(i+1)} \), i < j ≤ k − 2. By Claim 15 and (16), at least
\[
(1 − (k − 2)\sqrt{3}α')|A_{i(i+1)}| − (kα + α' + α'')\frac{n}{k} ≥ \frac{n}{2k}
\]
vertices can be selected as vi. After selecting v2, . . . , vk−2, we select two adjacent vertices vk−1 ∈ \( A_{k(i−1)} \) and vk ∈ \( A_k \) such that vk−1 and vk are in \( N(v2, · · · , vk−2) \). For j = k − 1, k, we know that \( N(vi) \) misses at most \( (kα + α' + α'')n/k \) vertices in \( A_j \) and at most \( (k−3)\sqrt{3}α'|A_j| \) vertices of \( A_j \) are not in \( N(v2, · · · , vk−2) \). Since \( d(A_{k−1}, A_k) > 6α'' \), at least
\[
6α''|A_{k−1}|,|A_k| − (kα + α' + α'')\frac{n}{k}(|A_{k−1}| + |A_k|)
\]
− 2(k−3)\( \sqrt{3}α'|A_{k−1}|,|A_k| ≥ 6\sqrt{3}α''\frac{n}{k}\)
pairs of vertices can be selected as vk−1, vk. Hence \( N(xv_1) \) contains at least
\[
(k−3)\frac{n}{2k} \sqrt{3}α''\frac{n}{k}(k−3)\frac{n}{2k} \sqrt{3}α''\frac{n}{k}\]
copies of \( K_{k−1} \). Let C be such a copy of \( K_{k−1} \). We follow the same procedure and construct a copy of \( K_{k−1} \) on \( N(yv_1) \setminus C \). After fixing k−3 sets \( A_{ij} \) with 2 ≤ i ≤ k − 2 and 3 ≤ j ≤ k − 1 such that no two of them are on the same row or column, still at least \( \frac{n}{2k} \) vertices can be selected as vi for 2 ≤ i ≤ k − 2. Then, as \( d(A_{ik}, A_{jk}) > 6α'' \), we can select at least \( 6\sqrt{3}α''\frac{n}{k} \) pairs of adjacent vertices \( v_{k−1} ∈ A_{(i−1)k} \) and \( v_k ∈ A_{ik} \) such that \( v_{k−1} \) and \( v_k \) are in \( N(v2, · · · , vk−2) \). This gives at least \( \sqrt{3}α''\frac{n}{k} \) copies of \( K_{k−1} \) in \( N(yv_1) \). Then, since at least \( |V| \cdot |A_{i1}| \cdot |A_{jk}| > \alpha \) vertices can be selected as vi, totally there are at least \( \alpha n\sqrt{3}α''\frac{n}{k} \) reachable \((2k−1)\)-sets from x to y, a contradiction.

Next we show that if any of \( d(A_{i1}, A_{i'1}) \) or \( d(A_{ik}, A_{i'k}) \), 2 ≤ i, i' ≤ k, is sufficiently large, then we can remove edges from G such that the resulting graph still satisfies the minimum degree condition, which contradicts the assumption that G is minimal.

**Claim 19.** For 2 ≤ i ̸= i' ≤ k, \( d(A_{i1}, A_{i'1}) \), \( d(A_{ik}, A_{i'k}) \), 2 ≤ i, i' ≤ k, is sufficiently large, then we can remove edges from G such that the resulting graph still satisfies the minimum degree condition, which contradicts the assumption that G is minimal.

**Proof.** Without loss of generality, assume that \( d(A_{2k}, A_{3k}) > 6k\sqrt{3}α'' \). By Claim 18, we have \( d(A_{2k}, A_{3k}) < 6α'' \). Combining with (14), we have \( d(A_{2j}, A_{3j}) < 6α'' \) for all \( j ∈ [k−1] \). Now fix \( j ∈ [k−1] \). The number of non-edges between \( A_{2j} \) and \( A_{3j} \) satisfies \( e(A_{2j}, A_{3j}) > (1 − 6α'')|A_{2j}|,|A_{3j}| \).

By the minimum degree condition and (15),
\[
e(A_{2k}, A_{3j}) < (1 + kα\frac{n}{k}|A_{3j}|) − (1 − 6α'')|A_{2j}|,|A_{3j}| ≤ 7α''\frac{n}{k}|A_{3j}|.
\]
Using (15) again, we obtain that
\[
d(A_{2k}, A_{3j}) ≥ 1 − 7α''\frac{n}{k}|A_{3j}| ≥ 1 − 8α''.
\]
Consequently \( d(A_{2k}, A_{3k}) ≥ 1 − 8α'' \), and at most \( \sqrt{8α''}|A_{2k}| \) vertices in \( A_{2k} \) are not \( \sqrt{8α''} \)-typical to \( A_{3k} \). Analogously at most \( \sqrt{8α''}|A_{3k}| \) vertices in \( A_{3k} \) are not \( \sqrt{8α''} \)-typical to \( A_{2k} \).

For \( \{i, j\} = \{2, 3\} \), define \( A_{Tik} \) as the set of the vertices in \( A_{ik} \) that are \( \sqrt{8α''} \)-typical to \( A_{jk} \). Let \( A_{Tik} = \{v ∈ A_{Tik} : \text{deg}_{A_{jk}}(v) ≤ \sqrt{8α''}|A_{jk}| \} \) and \( A_{Tik} = A_{ik} \setminus A_{Tik} \). For \( u ∈ A_{Tik} \), we have
\[
\text{deg}_{v_1}(u) = \text{deg}_{A_{jk}}(u) + \text{deg}_{A_{jk}}(u) > (1 − \sqrt{8α''})|A_{3k}| + \sqrt{8α''}|A_{3k}| = |A_{3k}|.
\]
Since \( |A_{3k}| = |v_2| \) is an integer, we have \( \text{deg}_{v_2}(u) ≥ \text{deg}_{v_2}(x) + 1 \). Similarly we can derive that \( \text{deg}_{v_2}(v) ≥ \text{deg}_{v_2}(x) + 1 \) for every \( v ∈ A_{Tik} \). If there is an edge uv joining some \( u ∈ A_{Tik} \) and some
v \in A_{3k}^2$, then we can delete this edge and the resulting graph still satisfies the minimum degree condition. Therefore we may assume that there is no edge between $A_{2k}^2$ and $A_{3k}^2$. Then
\[
e(A_{2k}, A_{3k}) \leq 2\sqrt{8\alpha''}|A_{2k}||A_{3k}| + |A_{2k}^2|\sqrt{8\alpha''}|A_{3k}^2| + |A_{3k}^2|\sqrt{8\alpha''}|A_{2k}^2|
\leq \sqrt{8\alpha''} (2|A_{2k}||A_{3k}| + |A_{2k}||A_{3k}^2| + |A_{3k}||A_{2k}^2|)
= \sqrt{8\alpha''} (|A_{2k}||V_3| + |A_{3k}||V_2|)
\leq 3\sqrt{\alpha''} \cdot 2k|A_{2k}||A_{3k}|.
\]
Therefore $d(A_{2k}, A_{3k}) \leq 6k\sqrt{\alpha''}$. 

In summary, by (15), (19) and (20), we have $(1 - 2\alpha'')\frac{n}{k} \leq |A_{ij}| \leq (1 + 2\alpha'')\frac{n}{k}$ for all $i$ and $j$. In order to make $\bigcup_{j=1}^{k-1} A_{ij}$ a partition of $V_1$, we move all the vertices of $A_{10}, A_{10}'$ to $A_{11}$. Since $|A_{10}|, |A_{10}'| \leq \alpha n$, the size of $A_{11}$ is increased by at most $2\alpha n$, thus we still have $|A_{ij}| - \frac{n}{k} \leq 2\alpha''\frac{n}{k}$. On the other hand, by (14), (18), and Claim 19, we have $d(A_{ij}, A_{i'j}) \leq 6k\sqrt{\alpha''}$ for $i \neq i'$ and all $j$ (at present $d(A_{11}, A_{11})\leq 2\alpha''$ for $i \geq 2$ because we added at most $2\alpha n$ vertices to $A_{11}$). Therefore after deleting edges, $G$ is $(2\alpha'', 6k\sqrt{\alpha''})$-approximate to $\Theta_{k \times k}(n/k)$. By (12), the definitions of $\alpha''$ and $\alpha'$, $G$ is $(\Delta/6, \Delta/2)$-approximate to $\Theta_{k \times k}(n/k)$.

\[\square\]

References

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(Jie Han and Yi Zhao) Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30303

E-mail address, Jie Han: jhan22@gsu.edu

E-mail address, Yi Zhao: yzhao6@gsu.edu