MINIMUM CODEGREE THRESHOLD FOR HAMILTON ℓ-CYCLES IN k-UNIFORM HYPERGRAPHS

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ABSTRACT. We show that for sufficiently large n, every k-uniform hypergraph on n vertices with minimum codegree at least \( \frac{n}{2(k - \ell)} \) contains a Hamilton ℓ-cycle. This codegree condition is best possible and improves on work of Hán and Schacht who proved an asymptotic result.

1. INTRODUCTION

A well-known result of Dirac [4] states that every graph G on \( n \geq 3 \) vertices with minimum degree \( \delta(G) \geq n/2 \) contains a Hamilton cycle. In recent years, researchers have extended this result to hypergraphs in various ways (see [17] for a survey). In order to state these results, we need to define degrees and Hamilton cycles for hypergraphs.

Given \( k \geq 2 \), a \( k \)-uniform hypergraph (in short, \( k \)-graph) consists of a vertex set \( V \) and an edge set \( E \subseteq \binom{V}{k} \), where every edge is a \( k \)-element subset of \( V \). Given a \( k \)-graph \( H \) with a set \( S \) of \( d \) vertices (where \( 1 \leq d \leq k - 1 \)) we define \( \deg_H(S) \) to be the number of edges containing \( S \) (the subscript \( H \) is omitted if it is clear from the context). The minimum \( d \)-degree \( \delta_d(H) \) of \( H \) is the minimum of \( \deg_H(S) \) over all \( d \)-vertex sets \( S \) in \( H \). We refer to \( \delta_1(H) \) as the minimum vertex degree and \( \delta_{k-1}(H) \) the minimum codegree of \( H \). For \( 1 \leq \ell < k \), a \( k \)-graph is a called an \( \ell \)-cycle if its vertices can be ordered cyclically such that each of its edges consists of \( k \) consecutive vertices and every two consecutive edges (in the natural order of the edges) share exactly \( \ell \) vertices. In \( k \)-graphs, a \( (k - 1) \)-cycle is often called a tight cycle while a 1-cycle is often called a loose cycle. We say that a \( k \)-graph contains a Hamilton \( \ell \)-cycle if it contains an \( \ell \)-cycle as a spanning subhypergraph. Note that a \( k \)-uniform \( \ell \)-cycle on \( n \) vertices contains exactly \( n/(k - \ell) \) edges, implying that \( k - \ell \) divides \( n \).

Confirming a conjecture of Katona and Kierstead [11], Rödl, Ruciński and Szemerédi [18, 19] showed that for any fixed \( k \), every \( k \)-graph \( H \) on \( n \) vertices with \( \delta_{k-1}(H) \geq n/2 + o(n) \) contains a tight Hamilton cycle. When \( k - \ell \) divides \( k \), a \((k - 1)\)-cycle on \( V \) trivially contains an \( \ell \)-cycle on \( V \) (provided \( k - \ell \) divides \( |V| \)). Thus the result in [19] implies that for all \( 1 \leq \ell < k \) such that \( k - \ell \) divides \( k \), every \( k \)-graph \( H \) on \( n \in (k - \ell)N \) vertices with \( \delta_{k-1}(H) \geq n/2 + o(n) \) contains a Hamilton \( \ell \)-cycle. It is not hard to see that these results are best possible up to the \( o(n) \) term. With long and involved arguments, Rödl, Ruciński and Szemerédi [20] determined the minimum codegree threshold for tight Hamilton cycles in 3-graphs.

Loose Hamilton cycles were first studied by Kühn and Osthus [14], who proved that every 3-graph on \( n \) vertices with \( \delta_2(H) \geq n/4 + o(n) \) contains a loose Hamilton cycle. It is easy to see that this is asymptotically best possible. It was generalized to arbitrary \( k \) by Keevash, Kühn, Mycroft, and Osthus [12] and to arbitrary \( k \) and arbitrary \( \ell < k/2 \) by Hán and Schacht [7].
Theorem 1.1. \cite{7} Fix integers \( k \geq 3 \) and \( 1 \leq \ell < k/2 \). Assume that \( \gamma > 0 \) and \( n \in (k-\ell)\mathbb{N} \) is sufficiently large. If \( H = (V, E) \) is a \( k \)-graph on \( n \) vertices such that \( \delta_{k-1}(H) \geq \left( \frac{1}{\ell(k-\ell)} + \gamma \right)n \), then \( H \) contains a Hamilton \( \ell \)-cycle.

Later Kühn, Mycroft, and Osthus \cite{13} proved that whenever \( k-\ell \) does not divide \( k \), every \( k \)-graph on \( n \) vertices with \( \delta_{k-1}(H) \geq \frac{n}{\ell(k-\ell)} + o(n) \) contains a Hamilton \( \ell \)-cycle. Since \( \lfloor k/(k-\ell) \rfloor = 2 \) when \( \ell < k/2 \), this generalizes Theorem 1.1 and is best possible up to the \( o(n) \) term. Recently Buss, Hán, and Schacht \cite{1} studied the minimum vertex degree condition and proved that every 3-graph \( H \) on \( n \) vertices with \( \delta_1(H) \geq \left( \frac{2}{16} + o(1) \right)\binom{n}{2} \) contains a loose Hamilton cycle. Recently we \cite{9} improved this to an exact result.

Rödl and Ruciński \cite[Problem 2.9]{17} asked for the exact minimum codegree threshold for Hamilton \( \ell \)-cycles in \( k \)-graphs. The \( k = 3 \) and \( \ell = 1 \) case was answered by Czygrinow and Molla \cite{3} recently. In this paper we determine this threshold for all \( k \geq 3 \) and \( \ell < k/2 \).

Theorem 1.2 (Main Result). Fix integers \( k \geq 3 \) and \( 1 \leq \ell < k/2 \). Assume that \( n \in (k-\ell)\mathbb{N} \) is sufficiently large. If \( H = (V, E) \) is a \( k \)-graph on \( n \) vertices such that

\[
\delta_{k-1}(H) \geq \frac{n}{2(k-\ell)},
\]

then \( H \) contains a Hamilton \( \ell \)-cycle.

A simple well-known construction shows that Theorem 1.2 is best possible – in fact, it works for all \( \ell < k \). Let \( H_0 = (V, E) \) be an \( n \)-vertex \( k \)-graph in which \( V \) is partitioned into sets \( A \) and \( B \) such that \( |A| = \left\lceil \frac{n}{\ell(k-\ell)} \right\rceil - 1 \). The edge set \( E \) consists of all \( k \)-sets that intersect \( A \). It is easy to see (e.g. \cite[Proposition 2.2]{13}) that \( \delta_{k-1}(H_0) = |A| \) and \( H_0 \) contains no Hamilton \( \ell \)-cycle.

Using the typical approach of obtaining exact results, our proof of Theorem 1.2 consists of an extremal case and a nonextremal case.

Definition 1.3. Let \( \Delta > 0 \), a \( k \)-graph \( H \) on \( n \) vertices is called \( \Delta \)-extremal if there is a set \( B \subseteq V(H) \), such that \( |B| = \left\lceil \frac{2(k-\ell)-1}{\ell(k-\ell)} \right\rceil n \) and \( e(B) \leq \Delta n^k \).

Theorem 1.4 (Nonextremal Case). For any integer \( k \geq 3 \), \( 1 \leq \ell < k/2 \) and \( 0 < \Delta < 1 \) there exists \( \gamma > 0 \) such that the following holds. Suppose that \( H \) is a \( k \)-graph on \( n \) vertices such that \( n \in (k-\ell)\mathbb{N} \) is sufficiently large. If \( H \) is not \( \Delta \)-extremal and satisfies \( \delta_{k-1}(H) \geq \left( \frac{1}{\ell(k-\ell)} - \gamma \right)n \), then \( H \) contains a Hamilton \( \ell \)-cycle.

Theorem 1.5 (Extremal Case). For any integer \( k \geq 3 \), \( 1 \leq \ell < k/2 \) there exists \( \Delta > 0 \) such that the following holds. Suppose \( H \) is a \( k \)-graph on \( n \) vertices such that \( n \in (k-\ell)\mathbb{N} \) is sufficiently large. If \( H \) is \( \Delta \)-extremal and satisfies \( (1.1) \), then \( H \) contains a Hamilton \( \ell \)-cycle.

Theorem 1.2 follows from Theorem 1.4 and 1.5 immediately by choosing \( \Delta \) from Theorem 1.5.

Let us compare our proof with those in aforementioned papers. There is no extremal case in \cite{7, 12, 13, 14} because only asymptotic results were proved. Our Theorem 1.5 is new and more general than \cite[Theorem 3.1]{3}. Following previous work \cite{18, 19, 20, 7, 13}, we prove Theorem 1.4 by using the absorbing method initiated by Rödl, Ruciński and Szemerédi. More precisely, we find the desired Hamilton \( \ell \)-cycle by applying the Absorbing Lemma (Lemma 2.1), the Reservoir Lemma (Lemma 2.2), and the Path-cover Lemma (Lemma 2.3). In fact, when \( \ell < k/2 \), the Absorbing Lemma and the Reservoir Lemma are not very difficult and already proven in \cite{7} (in contrast, when \( \ell > k/2 \), the Absorbing Lemma in \cite{13} is more difficult to prove). Thus the main step is to prove the Path-cover Lemma. As shown in \cite{7, 13}, after the Regularity Lemma is applied, it suffices to prove that the cluster \( k \)-graph \( K \) can be tiled almost perfectly by the \( k \)-graph \( F_{k, \ell} \), whose vertex set
consists of disjoint sets $A_1, \ldots, A_{a-1}, B$ of size $k - 1$, and whose edges are all the $k$-sets of the form $A_i \cup \{b\}$ for $i = 1, \ldots, a-1$ and all $b \in B$, where $a = \lceil \frac{k}{k-1} \rceil (k - \ell)$. In this paper we reduce the problem to tile $K$ with a much simpler $k$-graph $\mathcal{Y}_{k,2\ell}$, which consists of two edges sharing $2\ell$ vertices. Because of the simple structure of $\mathcal{Y}_{k,2\ell}$, we can easily find an almost perfect $\mathcal{Y}_{k,2\ell}$-tiling unless $K$ is in the extremal case (thus the original $k$-graph $H$ is in the extremal case). Interestingly $\mathcal{Y}_{3,2}$-tiling was studied in the very first paper [14] on loose Hamilton cycles but as a separate problem. Our recent paper [9] indeed used $\mathcal{Y}_{3,2}$-tiling as a tool to prove the corresponding path-cover lemma. On the other hand, the authors of [3] used a different approach (without the Regularity Lemma) to prove the Path-tiling Lemma (though they did not state such lemma explicitly).

The rest of the paper is organized as follows: we prove Theorem 1.4 in Section 2 and Theorem 1.5 in Section 3, and give concluding remarks in Section 4.

**Notation.** Given an integer $k \geq 0$, a $k$-set is a set with $k$ elements. For a set $X$, we denote by $\binom{X}{k}$ the family of all $k$-subsets of $X$. Given a $k$-graph $H$ and a set $A \subseteq V(H)$, we denote by $e_H(A)$ the number of the edges of $H$ in $A$. In this paper we often omit the subscript that represents the underlying hypergraph if it is clear from the context. Given a $k$-graph $H$ with two vertex sets $S, R$ such that $|S| < k$, we denote by $\deg_H(S, R)$ the number of $(k-|S|)$-sets $T \subseteq R$ such that $S \cup T$ is an edge of $H$ (in this case $T$ is called a neighbor of $S$). We define $\deg_H(S, R) = \binom{|V(H)|}{k-|S|} - \deg(S, R)$ as the number of non-edges on $S \cup R$ that contain $S$. When $R = V(H)$ (and $H$ is obvious), we simply write $\deg(S)$ and $\deg(S)$. When $S = \{v\}$, we use $\deg(v, R)$ instead of $\deg(S, R)$.

A $k$-graph $P$ is an $\ell$-path if there is an ordering $(v_1, \ldots, v_t)$ of its vertices such that every edge consists of $k$ consecutive vertices and two consecutive edges intersect in exactly $\ell$ vertices. Note that this implies that $k - \ell$ divides $t - \ell$. In this case we write $P = v_1 \cdots v_t$ and call two $\ell$-sets $\{v_1, \ldots, v_\ell\}$ and $\{v_{t-\ell+1}, \ldots, v_t\}$ ends of $P$.

## 2. Proof of Theorem 1.4

In this section we prove Theorem 1.4 by following the same approach as in [7].

### 2.1. Auxiliary lemmas and Proof of Theorem 1.4

We need [7, Lemma 5] and [7, Lemma 6] of H\'an and Schacht, in which any linear codegree is sufficient.

**Lemma 2.1** (Absorbing lemma,[7]). For all integers $k \geq 3$ and $1 \leq \ell < k/2$ and every $\gamma_1 > 0$ there exist $\eta > 0$ and an integer $n_0$ such that the following holds. Let $H$ be a $k$-graph on $n \geq n_0$ vertices with $\delta_{k-1}(H) \geq \gamma_1 n$. Then there is an $\ell$-path $P$ with $|V(P)| \leq \gamma_1^2 n$ such that for all subsets $U \subset V\setminus V(P)$ of size $|U| \leq \eta n$ and $|U| \in (k-\ell)\mathbb{N}$ there exists an $\ell$-path $Q \subset H$ with $V(Q) = V(P) \cup U$ such that $P$ and $Q$ have exactly the same ends (we say $P$ absorbs $U$ in this case).

**Lemma 2.2** (Reservoir lemma, [7]). For all integers $k \geq 3$ and $1 \leq \ell < k/2$ and every $d, \gamma_2 > 0$ there exists an $n_0$ such that the following holds. Let $H$ be a $k$-graph on $n > n_0$ vertices with $\delta_{k-1}(H) \geq dn$, then there is a set $R$ of size at most $\gamma_2 n$ such that for all $(k-1)$-sets $S \in \binom{V}{k-1}$ we have $\deg(S, R) \geq d \gamma_2 n/2$.

The main step in our proof of Theorem 1.4 is the following lemma, which is stronger than [7, Lemma 7].

**Lemma 2.3** (Path-cover lemma). For all integers $k \geq 3$, $1 \leq \ell < k/2$, and every $\gamma_3, \alpha > 0$ there exist integers $p$ and $n_0$ such that the following holds. Let $H$ be a $k$-graph on $n > n_0$ vertices with $\delta_{k-1}(H) \geq \frac{1}{2\gamma_3} - \gamma_3 \alpha n$, then there is a family of at most $p$ vertex disjoint $\ell$-paths that together cover all but at most $\alpha n$ vertices of $H$, or $H$ is $14\gamma_3$-extremal.

We can now prove Theorem 1.4 in a similar way as in [7].
Proof of Theorem 1.4. Given \( k \geq 3, 1 \leq \ell < k/2 \) and \( 0 < \Delta < 1 \), let \( \gamma = \min\left\{ \frac{\Delta}{3}, \frac{1}{4k^2} \right\} \) and \( n \in (k - \ell)N \) be sufficiently large. Suppose that \( \mathcal{H} = (V, E) \) is a \( k \)-graph on \( n \) vertices with \( \delta_{k-1}(\mathcal{H}) \geq (\frac{1}{2(k-\ell)} - \gamma) n \). Since \( \frac{1}{2(k-\ell)} - \gamma > \gamma \), we can apply Lemma 2.1 with \( \gamma_1 = \gamma \) and obtain \( \eta > 0 \) and an absorbing path \( \mathcal{P}_0 \) with ends \( S_0, T_0 \) such that \( \mathcal{P}_0 \) absorbs any \( u \) vertices outside \( \mathcal{P}_0 \) if \( u \leq \eta n \) and \( u \in (k - \ell)N \).

Let \( V_1 = (V \setminus V(\mathcal{P}_0)) \cup S_0 \cup T_0 \) and \( \mathcal{H}_1 = \mathcal{H}[V_1] \). Note that \( |V(\mathcal{P}_0)| \leq \gamma^5 n \) implies that \( \delta_{k-1}(\mathcal{H}_1) \geq (\frac{1}{2(k-\ell)} - \gamma) n - \gamma^5 n \leq \frac{1}{2} n \) since \( \gamma < \frac{1}{4k^2} \) and \( \ell \geq 1 \). We next apply Lemma 2.2 with \( d = \frac{1}{2k} \) and \( \gamma_2 = \min\{\eta/2, \gamma\} \) to \( \mathcal{H}_1 \) and get a reservoir \( R \subset V_1 \) such that for any \((k-1)\)set \( S \subset V_1 \), we have

\[
\deg(S, R) \geq d \gamma_2 |V_1|/2 \geq d \gamma_2 n/4. \tag{2.1}
\]

Let \( V_2 = V \setminus (V(\mathcal{P}_0) \cup R) \), \( n_2 = |V_2| \), and \( \mathcal{H}_2 = \mathcal{H}[V_2] \). Note that \( |V(\mathcal{P}_0) \cup R| \leq \gamma_1 n + \gamma_2 n \leq 2 \gamma n \), so

\[
\delta_{k-1}(\mathcal{H}_2) \geq \left( \frac{1}{2(k-\ell)} - \gamma \right) n - 2 \gamma n \geq \left( \frac{1}{2(k-\ell)} - 3 \gamma \right) n_2.
\]

Applying Lemma 2.3 to \( \mathcal{H}_2 \) with \( \gamma_3 = 3 \gamma \) and \( \alpha = \eta/2 \), we obtain at most \( p \) vertex disjoint \( \ell \)-paths that cover all but at most \( \alpha n_2 \) vertices of \( \mathcal{H}_2 \), unless \( \mathcal{H}_2 \) is 14\( \gamma_3 \)-extremal. In the latter case, there exists \( B' \subset V_2 \) such that \( |B'| = \lfloor \frac{2k-2\ell-1}{2(k-\ell)} n_2 \rfloor \) and \( e(B') \leq 42 \gamma n_2^2 \). Then we add at most \( n - n_2 \leq 2 \gamma n \) vertices from \( V \setminus B' \) to \( B' \) and obtain a vertex set \( B \subset V(\mathcal{H}) \) such that \( |B| = \lfloor \frac{2k-2\ell-1}{2(k-\ell)} n \rfloor \) and

\[
e(B) \leq 42 \gamma n_2^2 + 2 \gamma n \cdot \frac{n - 1}{k - 1} \leq 42 \gamma n^k + \gamma n^k \leq \Delta n^k,
\]

which means that \( \mathcal{H} \) is \( \Delta \)-extremal, a contradiction. In the former case, denote these \( \ell \)-paths by \( \{\mathcal{P}_i\}_{i \in [p']} \) for some \( p' \leq p \), and their ends by \( \{S_i, T_i\}_{i \in [p']} \). Note that both \( S_i \) and \( T_i \) are \( \ell \)-sets for \( \ell < k/2 \). We arbitrarily pick disjoint \((k-2\ell-1)\)sets \( X_0, X_1, \ldots, X_{p'} \subset R \setminus (S_0 \cup T_0) \) (note that \( k - 2\ell - 1 \geq 0 \)). Let \( T_{p'+1} = T_0 \). By (2.1), we get for \( 0 \leq i \leq p' \),

\[
\deg \left( S_i \cup T_{i+1} \cup X_i, R \setminus \bigcup_{0 \leq i \leq p'} (S_i \cup T_i \cup X_i) \right) \geq d \gamma_2 n/4 - (p' + 1)(k - 1) \geq p + 1,
\]
as \( n \) is large enough. So we can connect \( \mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_{p'} \) by using vertices from \( R \) and get an \( \ell \)-cycle \( \mathcal{C} \). Note that \( |V(\mathcal{H}) \setminus V(\mathcal{C})| \leq |R| + \alpha n_2 \leq \gamma \eta n + \alpha n \leq \eta n \) and since \( n \in (k - \ell)N \), \( |V \setminus V(\mathcal{C})| \) is also a multiple of \( k - \ell \). So we can use \( \mathcal{P}_0 \) to absorb all unused vertices in \( R \) and uncovered vertices in \( V_2 \) thus obtaining a Hamilton \( \ell \)-cycle in \( \mathcal{H} \).

The rest of this section is devoted to the proof of Lemma 2.3.

2.2. Proof of Lemma 2.3. Following the approach in [7], we use the Weak Regularity Lemma, which is a straightforward extension of Szemerédi’s regularity lemma for graphs [21].

Let \( \mathcal{H} = (V, E) \) be a \( k \)-graph and let \( A_1, \ldots, A_k \) be mutually disjoint non-empty subsets of \( V \). We define \( e(A_1, \ldots, A_k) \) to be the number of edges with one vertex in each \( A_i, i \in [k] \), and the density of \( \mathcal{H} \) with respect to \( (A_1, \ldots, A_k) \) as

\[
d(A_1, \ldots, A_k) = \frac{e(A_1, \ldots, A_k)}{|A_1| \cdots |A_k|}.
\]

We say a \( k \)-tuple \((V_1, \ldots, V_k)\) of mutually disjoint subsets \( V_1, \ldots, V_k \subset V \) is \((\epsilon, d)\)-regular, for \( \epsilon > 0 \) and \( d \geq 0 \), if

\[
|d(A_1, \ldots, A_k) - d| \leq \epsilon
\]
for all \( k \)-tuples of subsets \( A_i \subset V_i, i \in [k] \), satisfying \( |A_i| \geq \epsilon |V_i| \). We say \((V_1, \ldots, V_k)\) is \( \epsilon \)-regular if it is \((\epsilon, d)\)-regular for some \( d \geq 0 \). It is immediate from the definition that in an \((\epsilon, d)\)-regular \( k \)-tuple
(V_1, \ldots, V_k)$, if $V'_i \subset V_i$ has size $|V'_i| \geq c|V_i|$ for some $c \geq c$, then $(V'_1, \ldots, V'_k)$ is $(\max\{\epsilon/c, 2\epsilon\}, d)$-regular.

**Theorem 2.4** (Weak Regularity Lemma). Given $t_0 \geq 0$ and $\epsilon > 0$, there exist $T_0 = T_0(t_0, \epsilon)$ and $n_0 = n_0(t_0, \epsilon)$ so that for every $k$-graph $H = (V, E)$ on $n > n_0$ vertices, there exists a partition $V = V_0 \cup V_1 \cup \cdots \cup V_t$ such that

(i) $t_0 \leq t \leq T_0$,

(ii) $|V_i| = |V_2| = \cdots = |V_t|$ and $|V_0| \leq cn$,

(iii) for all but at most $\epsilon|\binom{k}{3}|$ $k$-subsets $\{i_1, \ldots, i_k\} \subset [t]$, the $k$-tuple $(V_{i_1}, \ldots, V_{i_k})$ is $\epsilon$-regular.

The partition given in Theorem 2.4 is called an $\epsilon$-regular partition of $H$. Given an $\epsilon$-regular partition of $H$ and $d \geq 0$, we refer to $V_i, i \in [t]$ as clusters and define the cluster hypergraph $K = K(\epsilon, d)$ with vertex set $[t]$ and $\{i_1, \ldots, i_k\} \subset [t]$ is an edge if and only if $(V_{i_1}, \ldots, V_{i_k})$ is $\epsilon$-regular and $d(V_{i_1}, \ldots, V_{i_k}) \geq d$.

The following corollary shows that the cluster hypergraph inherits the minimum degree of the original hypergraph. Its proof is almost the same as in [7, Proposition 16] after we replace $\frac{1}{2(k-\ell)} + \gamma$ by $c - \gamma$ we thus omit the proof.

**Corollary 2.5.** [7] Given $c, \epsilon, d > 0$ and integers $k \geq 3, t_0$ such that $0 < \epsilon < d^2/4$ and $t_0 \geq 2k/d$, there exist $T_0$ and $n_0$ such that the following holds. Let $H$ be a $k$-graph on $n > n_0$ vertices such that $\delta_{k-1}(H) \geq cn$. If $H$ has an $\epsilon$-regular partition $V_0 \cup V_1 \cup \cdots \cup V_t$ with $t_0 \leq t \leq T_0$ and $K = K(\epsilon, d)$ is the cluster hypergraph, then at most $\sqrt{\epsilon}^{k-1} (k-1)$-subsets $S$ of $[t]$ violate $\deg_K(S) \geq (c - 2d)t$.

Let $H$ be a $k$-partite $k$-graph with partition classes $V_1, \ldots, V_k$. Then we call an $\ell$-path $P$ of $H$ with edges $\{E_1, \ldots, E_t\}$ canonical with respect to $(V_1, \ldots, V_k)$ if

$$E_i \cap E_{i+1} \subseteq \bigcup_{j \in [t]} V_j \quad \text{or} \quad E_i \cap E_{i+1} \subseteq \bigcup_{j \in [2\ell]\setminus[t]} V_j$$

for $i = 1, \ldots, t-1$. Note that a canonical $\ell$-path with an odd length $t$ contains $\frac{t+1}{2}$ vertices of $V_i$ for $i \in [2\ell]$ and $t$ vertices of $V_i$ for $i > 2\ell$.

We also need the following proposition from [7].

**Proposition 2.6.** [7, Proposition 19] Suppose $H$ is a $k$-partite, $k$-graph with partition classes $V_1, \ldots, V_k$, $|V_i| = m$ for all $i \in [k]$, and $|E(H)| \geq dm^k$. Then there exists a canonical $\ell$-path in $H$ with $t > \frac{dm}{\ell^{k-\ell}}$ edges.

In [7] the authors used Proposition 2.6 to cover an $(\epsilon, d)$-regular tuple $(V_1, \ldots, V_k)$ of sizes $|V_1| = \cdots = |V_{k-1}| = (2k - 2\ell - 1)m$ and $|V_k| = (k - 1)m$ with vertex disjoint $\ell$-paths. Our next lemma shows that an $(\epsilon, d)$-regular tuple $(V_1, \ldots, V_k)$ of sizes $|V_1| = \cdots = |V_{2\ell}| = m$ and $|V_i| = 2m$ for $i > 2\ell$ can be covered with $\ell$-paths.

**Lemma 2.7.** Fix $k \geq 3, 1 \leq \ell < k/2$ and $\epsilon, d > 0$ such that $d > 2\epsilon$. Let $m > \frac{2\ell^2}{(d-\epsilon)e}$. Suppose $\mathcal{V} = (V_1, V_2, \ldots, V_k)$ is an $(\epsilon, d)$-regular $k$-tuple with

$$|V_1| = \cdots = |V_{2\ell}| = m \quad \text{and} \quad |V_{2\ell+1}| = \cdots = |V_k| = 2m. \quad (2.2)$$

Then there are at most $\frac{4(k-\ell)}{(d-\epsilon)e}$ vertex disjoint $\ell$-paths that together cover all but at most $2kem$ vertices of $\mathcal{V}$.

**Proof.** We greedily find disjoint canonical $\ell$-paths of odd length by Proposition 2.6 in $\mathcal{V}$ until less than $\epsilon m$ vertices are uncovered in $V_1$. Suppose that we have obtained odd $\ell$-paths $\mathcal{P}_1, \ldots, \mathcal{P}_p$ by Proposition 2.6 for some $p \geq 0$. Let $t = \sum_{j=1}^p e(\mathcal{P}_j)$. Since each $e(\mathcal{P}_j)$ are odd, $\bigcup_{j=1}^p \mathcal{P}_i$ contains $\frac{t+\epsilon p}{2}$
vertices of \( V_i \) for \( i \in [2\ell] \) and \( t \) vertices of \( V_i \) for \( i > 2\ell \). For \( i \in [k] \), let \( U_i \) be the set of uncovered vertices of \( V_i \) and assume that \( |U_1| \geq \epsilon m \). Using (2.2), we derive that \( |U_1| = \cdots = |U_{2\ell}| \geq \epsilon m \) and
\[
|U_{2\ell+1}| = \cdots = |U_k| = 2|U_1| + p. \tag{2.3}
\]
We pick an arbitrary \( k \)-partite subhypergraph \( \mathcal{V}' \) with \( |U_1| \) vertices in each \( U_i \) for \( i \in [k] \). By regularity, \( \mathcal{V}' \) contains at least \((d-\epsilon)|U_1|^{k}\) edges so that we can apply Proposition 2.6 and find an \( \ell \)-path of odd length at least \( (d-\epsilon)m\) - 1 (dismiss one edge if needed). We continue this process until \( |U_1| < \epsilon m \). Let \( \mathcal{P}_1, \ldots, \mathcal{P}_p \) be the \( \ell \)-paths obtained in \( \mathcal{V} \) after the iteration stops. Since \(|V_1 \cap V(\mathcal{P}_j)| \geq \frac{(d-\epsilon)m}{4k(k-\ell)}\) for every \( j \), we have
\[
p \leq \frac{m}{(d-\epsilon)m} = \frac{4(k-\ell)}{(d-\epsilon)\ell}.
\]
Since \( m > \frac{2k^2}{\epsilon t(d-\ell)} \), we further have
\[
p(k-2\ell) \leq \frac{4(k-\ell)(k-2\ell)}{(d-\epsilon)\ell} < \frac{4k^2}{(d-\epsilon)\ell} < 2\epsilon m.
\]
By (2.3), the total number of uncovered vertices in \( \mathcal{V} \) is
\[
\sum_{i=1}^{k} |U_i| = |U_1|2\ell + (2|U_1| + p)(k-2\ell) = 2(k-\ell)|U_1| + p(k-2\ell)
< 2(k-1)\epsilon m + 2\epsilon m = 2k\epsilon m.
\]
\[\square\]
Given \( k \geq 3 \) and \( 1 \leq b < k \), recall that \( \mathcal{Y}_{k,b} \) is a \( k \)-graph with two edges that share exactly \( b \) vertices. In general, given two (hyper)graphs \( \mathcal{G} \) and \( \mathcal{H} \), a \( \mathcal{G} \)-tiling is a sub(hyper)graph of \( \mathcal{H} \) that consists of vertex-disjoint copies of \( \mathcal{G} \). A \( \mathcal{G} \)-tiling is perfect if it is a spanning sub(hyper)graph of \( \mathcal{H} \). The following lemma is the main step in our proof of Lemma 2.3 and we prove it in the next subsection. Note that it generalizes [2, Lemma 3.1] of Czygrinow, DeBiasio, and Nagle.

**Lemma 2.8 (\( \mathcal{Y}_{k,b} \)-tiling Lemma).** Given integers \( k \geq 3 \), \( 1 \leq b < k \) and constants \( \gamma, \beta > 0 \), there exist \( 0 < \epsilon' < \gamma \beta \) and an integer \( n_0 \) such that the following holds. Suppose \( \mathcal{H} \) is a \( k \)-graph on \( n > n_0 \) vertices with \( \text{deg}(S) \geq \left( \frac{1}{2k-b} - \gamma \right) n \) for all but at most \( \epsilon' n^{k-1} \) sets \( S \in \binom{V}{k-1} \), then there is a \( \mathcal{Y}_{k,b} \)-tiling that covers all but at most \( \beta n \) vertices of \( \mathcal{H} \) unless \( \mathcal{H} \) contains a vertex set \( B \) such that \( |B| = \left[ \frac{2k-b-1}{2k-b} \right] n \) and \( \epsilon(\mathcal{H}) < 6\gamma \beta n^k \).

Now we are ready to prove Lemma 2.3.

**Proof of Lemma 2.3.** Fix such integers \( k, \ell \), \( 0 < \gamma, \alpha < 1 \). Let \( \epsilon' \) be the constant returned from Lemma 2.8 with \( b = 2\ell \), \( \gamma = 2\gamma_3 \), and \( \beta = \alpha/2 \). So \( \epsilon' < \gamma \beta = \gamma_3 \alpha \). Furthermore, let \( p = \frac{4\gamma_3}{(d-\ell)\epsilon} \), where \( T_0 \) is the constant returned from Corollary 2.5 with \( c = \frac{1}{2k-2\ell} - \gamma_3 \), \( \epsilon = (\epsilon')^2/16 \), and \( d = \gamma_3/2 \).

Let \( n \) be a sufficiently large integer and let \( \mathcal{H} \) be a \( k \)-graph on \( n \) vertices with \( \delta_{k-1}(\mathcal{H}) \geq \left( \frac{1}{2k-2\ell} - \gamma_3 \right) n \). By applying Corollary 2.5 with the constants chosen above we obtain an \( \epsilon \)-regular partition and a cluster hypergraph \( \mathcal{K} = \mathcal{K}(\epsilon, d) \) such that for all but at most \( \sqrt{\epsilon} t^{k-1} (k-1) \)-sets \( S \in \binom{V}{k-1} \),
\[
\text{deg}(\mathcal{K})(S) \geq \left( \frac{1}{2(k-\ell)} - \gamma_3 - 2d \right) t = \left( \frac{1}{2(k-\ell)} - 2\gamma_3 \right) t,
\]
because \( d = \gamma_3/2 \). Let \( m \) be the size of each cluster except \( V_0 \), then \((1-\epsilon)^2 \leq m \leq p \). Applying Lemma 2.8 with the constants chosen above, we derive that either there is a \( \mathcal{Y}_{k,2\ell} \)-tiling \( \mathcal{V}' \) of \( \mathcal{K} \) which covers all but at most \( \beta t \) vertices of \( \mathcal{K} \) or there exists a set \( B \subseteq V(\mathcal{K}) \), such that \(|B| = \left[ \frac{2k-2\ell-1}{2(k-\ell)} t \right] \).
and \(e_K(B) \leq 12\gamma_3 t^k\). In the latter case, let \(B' \subseteq V(H)\) be the union of the clusters in \(B\). By regularity,
\[
e_H(B') \leq e_K(B) \cdot m^k + \left(\frac{t}{k}\right) \cdot d \cdot m^k + \epsilon \cdot \left(\frac{t}{k}\right) \cdot m^k + \left(\frac{m}{2}\right) \left(\frac{n}{k-2}\right),
\]
where the right-hand side bounds the number of edges from regular \(k\)-tuples with high density, edges from regular \(k\)-tuples with low density, edges from irregular \(k\)-tuples and edges that lie in at most \(k-1\) clusters. Since \(m \leq \frac{n}{3}, \epsilon < \gamma_3, d = \gamma_3/2, \) and \(t^{-2} < t_0^2 < \gamma_3\), we obtain that
\[
e_H(B') \leq 12\gamma_3 t^k \cdot \left(\frac{n}{k}\right)^k + \left(\frac{t}{k}\right) \cdot d \cdot \left(\frac{n}{k}\right)^k + \epsilon \left(\frac{t}{k}\right) \left(\frac{n}{k}\right)^k + \left(\frac{m}{2}\right) \left(\frac{n}{k-2}\right) \leq 13\gamma_3 n^k.
\]

Note that \(|B'| = \left\lfloor \frac{k-2t-1}{2(k-\ell)} \right\rfloor m \leq \frac{k-2t-1}{2(k-\ell)} \cdot \frac{n}{\ell} = \frac{k-2t-1}{2(k-\ell)} n\), and consequently \(|B'| \leq \left\lfloor \frac{k-2t-1}{2(k-\ell)} n\right\rfloor\). On the other hand,
\[
|B'| = \left\lfloor \frac{k-2t-1}{2(k-\ell)} \right\rfloor m \geq \left(\frac{k-2t-1}{2(k-\ell)} - 1\right) \left(1 - \frac{n}{\ell}\right) \geq \left(\frac{k-2t-1}{2(k-\ell)} - \epsilon\right) \frac{n}{\ell} = \frac{k-2t-1}{2(k-\ell)} n - \epsilon n.
\]

By adding at most \(en\) vertices from \(V \setminus B' \) to \(B'\), we get a set \(B'' \subseteq V(H)\) of size exactly \(\left\lfloor \frac{k-2t-1}{2(k-\ell)} n\right\rfloor\), with \(e(B'') \leq e(B') + \epsilon n \cdot n^{k-1} < 14\gamma_3 n^k\). Hence \(H\) is \(14\gamma_3\)-extremal.

In the former case, the union of the clusters covered by \(\mathcal{Y}\) contains all but at most \(\beta tn + |V_0| \leq \alpha n/2 + \epsilon n\) vertices. We apply Lemma 2.7 to each member \(Y' \in \mathcal{Y}\). Suppose that \(Y'\) has the vertex set \([2k-2\ell]\) with edges \(\{1, \ldots, k\}\) and \(\{k-2\ell+1, \ldots, 2k-2\ell\}\). For \(i \in [2k-2\ell], \) let \(W_i\) denote the corresponding cluster in \(H\). We split each \(W_i, \) \(i = k-2\ell+1, \ldots, k, \) into two disjoint sets \(W_i^1\) and \(W_i^2\) of equal size. Then the \(k\)-tuples \((W_{k-2\ell+1}^1, \ldots, W_{k}^1, W_{k-2\ell}, \ldots, W_{k+1}^2)\) and \((W_{k-2\ell+1}^2, \ldots, W_{k}^2, W_{k+1}, \ldots, W_{k-2\ell})\) are \((2, d)\)-regular and of sizes \(2\ell / 2\), \(2\ell / 2, m, \ldots, m.\) Applying Lemma 2.7 to these two \(k\)-tuples with \(m' = \frac{2\ell}{2}\), we find a family of disjoint loose paths in each \(k\)-tuple covering all but at most \(2k\ell m' = k\ell m\) vertices.

Since \(|\mathcal{Y}| \leq \frac{t}{2k-2\ell}, \) we obtain a path-tiling that consists of at most \(2k\ell \cdot \frac{4(k-\ell)}{2(k-\ell)} \leq \frac{4\gamma_3}{d-\epsilon} \) paths and covers all but at most
\[
2k\ell m' \cdot \frac{t}{2k-2\ell} + \alpha n/2 + \epsilon n < 3\alpha n + \alpha n < 3\alpha n
\]
vertices, where we use \(2k - 2\ell > k\) and \(\epsilon = (\epsilon')^2/16 < (\gamma_3^2)/16 < \alpha/6\). This completes the proof.

\[\square\]

2.3. Proof of Lemma 2.8. We first give an upper bound on the size of \(k\)-graphs containing no copy of \(Y_{k,b}\). In its proof, we use the concept of link (hyper)graph: given a \(k\)-graph \(H\) with a set \(S\) of at most \(k-1\) vertices, the link graph of \(S\) is the \((k-|S|)\)-graph with vertex set \(V(H) \setminus S\) and edge set \(\{e \setminus S : e \in E(H), S \subseteq e\}\). Throughout the rest of the paper, we frequently use the simple identity \(\binom{m+b}{k-b} = \binom{m}{k} \binom{k}{b}\), which holds for all integers \(1 \leq b \leq k \leq m\).

Fact 2.9. Let \(1 \leq b < k \leq m\). If \(H\) is a \(k\)-graph on \(m\) vertices containing no copy of \(Y_{k,b}\), then \(e(H) < \binom{m}{k-1}\).
Proof. Fix any $b$-set $S \subseteq V(\mathcal{H})$ and consider its link graph $L_S$. Since $\mathcal{H}$ contains no copy of $\mathcal{Y}_{k,b}$, any two edges of $L_S$ intersect. By the Erdős–Ko–Rado Theorem [5], $|L_S| \leq \binom{m-b-1}{k-b-1}$. Thus,

$e(\mathcal{H}) \leq \frac{1}{k} \binom{m}{b} \cdot \binom{m-b-1}{k-b-1} = \frac{1}{k} \binom{m}{b} \binom{m-b}{k-b} \frac{k-b}{m-b} = \binom{m}{k} \frac{k-b}{m-b} \leq \binom{m}{k-1} \frac{k-b}{m-b} < \binom{m}{k-1}.$

\[ \square \]

Proof of Lemma 2.8. Given $\gamma, \beta > 0$, let $\epsilon' = \frac{\gamma \beta^{k-1}}{(k-1)!}$ and $n \in \mathbb{N}$ be sufficiently large. Let $\mathcal{H}$ be a $k$-graph on $n$ vertices that satisfies $\text{deg}(S) \geq \left( \frac{1}{2k-2} - \gamma \right)n$ for all but at most $\epsilon' n^{k-1} (k-1)$-sets $S$. Fix a largest $\mathcal{Y}_{k,b}$-tiling $\mathcal{Y} = \{\mathcal{Y}_1, \ldots, \mathcal{Y}_m\}$ and write $V_i = V(\mathcal{Y}_i)$ for $i \in [m]$. Let $V' = \bigcup_{i \in [m]} V_i$ and $U = V(\mathcal{H}) \setminus V'$. Assume that $|U| > \beta n$ otherwise we are done.

Let $C$ be the set of vertices $v \in V'$ such that $\text{deg}(v, U) \geq (2k-b)^2 \binom{|U|}{k-2}$. We will show that $|C| \leq \frac{n}{2k-2}$ and $C$ covers almost all the edges of $\mathcal{H}$, which implies that $\mathcal{H}[V \setminus C]$ is sparse and $\mathcal{H}$ is in the extremal case. We first observe that every $\mathcal{Y}_i \in \mathcal{Y}$ contains at most one vertex in $C$. Suppose instead, two vertices $x, y \in V_i$ are both in $C$. Since $\text{deg}(x, U) \geq (2k-b)^2 \binom{|U|}{k-2}$, by Fact 2.9, there is a copy of $\mathcal{Y}_{k-1,b}$ in the link graph of $x$ on $U$, which gives rise to $\mathcal{Y}'$, a copy of $\mathcal{Y}_{k,b}$ on $\{x\} \cup U$. Since the link graph of $y$ on $U \setminus V(\mathcal{Y}')$ has at least

$$(2k-b)^2 \binom{|U|}{k-2} - (2k-b-1) \binom{|U|}{k-2} + \binom{|U| \setminus V(\mathcal{Y}')}{k-2}$$

edges, we can find another copy of $\mathcal{Y}_{k,b}$ on $\{y\} \cup (U \setminus V(\mathcal{Y}'))$ by Fact 2.9. Replacing $\mathcal{Y}_i$ in $\mathcal{Y}$ with these two copies of $\mathcal{Y}_{k,b}$ creates a $\mathcal{Y}_{k,b}$-tiling larger than $\mathcal{Y}$, contradiction. Consequently,

$$\sum_{S \in \binom{V'}{k-1}} \text{deg}(S, V') \leq |C| \binom{|U|}{k-1} + |V' \setminus C| (2k-b)^2 \binom{|U|}{k-2}$$

$$\leq |C| \binom{|U|}{k-1} + (2k-b)^2 n \binom{|U|}{k-2} \quad \text{because } |V' \setminus C| < n$$

$$= \binom{|U|}{k-1} \left( |C| + \frac{(2k-b)^2 n (k-1)}{|U| - k + 2} \right).$$

(2.4)

Second, by Fact 2.9, $e(U) \leq \left( \binom{|U|}{k-1} \right)$ since $\mathcal{H}[U]$ contains no copy of $\mathcal{Y}_{k,b}$, which implies

$$\sum_{S \in \binom{V}{k-1}} \text{deg}(S, U) \leq k \binom{|U|}{k-1}.$$

(2.5)

By the definition of $\epsilon'$, we have

$$\epsilon' n^{k-1} = \frac{\gamma \beta^{k-1}}{(k-1)!} n^{k-1} < \frac{\gamma |U|^{k-1}}{(k-1)!} < 2\gamma \binom{|U|}{k-1},$$

since $|U|$ is large enough. At last, by the degree condition, we have

$$\sum_{S \in \binom{V}{k-1}} \text{deg}(S) \geq \left( \binom{|U|}{k-1} - \epsilon' n^{k-1} \right) \left( \frac{1}{2k-b} - \gamma \right) n > (1 - 2\gamma) \binom{|U|}{k-1} \left( \frac{1}{2k-b} - \gamma \right) n.$$

(2.6)

Since $\text{deg}(S) = \text{deg}(S, U) + \text{deg}(S, V')$, we combine (2.4), (2.5) and (2.6) and get

$$|C| > (1 - 2\gamma) \left( \frac{1}{2k-b} - \gamma \right) n - \frac{(2k-b)^2 n (k-1)}{|U| - k + 2}.$$
Since $|U| > 16k^3/\gamma$, we get
\[
\frac{(2k-b)^2n(k-1)}{|U| - k + 2} < \frac{4k^3n}{|U|/2} < \gamma n/2.
\]
As $2\gamma^2n > k$ and $2k-b \geq 4$, it follows that $|C| > \left(\frac{1}{2k-b} - 2\gamma\right)n$.

Let $I_C$ be the set of all $i \in [m]$ such that $V_i \cap C \neq \emptyset$. Since each $V_i$, $i \in I_C$, contains one vertex of $C$, we have
\[
|I_C| = |C| \geq \left(\frac{1}{2k-b} - 2\gamma\right)n \geq m - 2\gamma n.
\]  
(2.7)

Let $A = (\bigcup_{i \in I_C} V_i \setminus C) \cup U$.

**Claim 2.10.** $\mathcal{H}(A)$ contains no copy of $\mathcal{Y}_{k,b}$, thus $e(A) \leq \binom{n}{k-1}$.

**Proof.** The first half of the claim implies the second half by Fact 2.9. Suppose instead, $\mathcal{H}(A)$ contains a copy of $\mathcal{Y}_{k,b}$, denoted by $Y_0$. Note that $V(Y_0) \not\subseteq U$ because $\mathcal{H}(U)$ contains no copy of $\mathcal{Y}_{k,b}$. Without loss of generality, suppose that $V_1, \ldots, V_j$ contain the vertices of $Y_0$ for some $j \leq 2k-b$. For $i \in [j]$, let $c_i$ denote the unique vertex in $V_i \cap C$. We greedily construct vertex-disjoint copies of $\mathcal{Y}_{k,b}$ on $\{c_i\} \cup U$, $i \in [j]$ as follows. Suppose we have found $Y_1', \ldots, Y_j'$ (copies of $\mathcal{Y}_{k,b}$) for some $i < j$. Let $U_0$ denote the set of the vertices of $U$ covered by $Y_0, Y_1', \ldots, Y_j'$. Then $|U_0| \leq (i+1)(2k-b-1) \leq (2k-b)(2k-b-1)$.

Since $\deg(c_{i+1}, U) \geq (2k-b)^2\left(\frac{|U|}{k-2}\right)$, the link graph of $c_{i+1}$ on $U \setminus U_0$ has at least
\[
(2k-b)^2\left(\frac{|U|}{k-2}\right) - |U_0|\left(\frac{|U|}{k-2}\right) > \left(\frac{|U|}{k-2}\right)
\]
edges. By Fact 2.9, there is a copy of $\mathcal{Y}_{k,b}$ on $\{c_{i+1}\} \cup (U \setminus U_0)$. Let $Y_1'', \ldots, Y_j''$ denote the copies of $\mathcal{Y}_{k,b}$ constructed in this way. Replacing $Y_1, \ldots, Y_j$ in $\mathcal{Y}$ with $Y_0, Y_1'', \ldots, Y_j''$ gives a $\mathcal{Y}_{k,b}$-tiling larger than $\mathcal{Y}$, contradiction.

Note that the edges not incident to $C$ are either contained in $A$ or intersect some $V_i$, $i \notin I_C$. By (2.7) and Claim 2.10,
\[
eq V \setminus C \leq e(A) + (2k-b) \cdot 2\gamma n \left(\frac{n-1}{k-1}\right) < \left(\frac{n}{k-1}\right) + (4k-2b)\gamma n \left(\frac{n}{k-1}\right)
\]
\[
< 4k\gamma n \left(\frac{n}{k-1}\right) = \frac{4k}{(k-1)^2} \cdot \gamma n^k \leq 6\gamma n^k,
\]
where the last inequality follows from $k \geq 3$. Since $|C| \leq \frac{n}{2k-b}$, we can pick a set $B \subseteq V \setminus C$ of order $[\frac{2k-b-1}{2k-b}]n$ such that $e(B) < 6\gamma n^k$. \hfill \Box

### 3. The Extremal Theorem

In this section we prove Theorem 1.5. Assume that $k \geq 3$, $1 \leq \ell < k/2$ and $0 < \Delta \ll 1$. Let $n \in (k-\ell)\mathbb{N}$ be sufficiently large. Let $\mathcal{H}$ be a $k$-graph on $V$ of $n$ vertices such that $\delta_{k-1}(\mathcal{H}) \geq \frac{n}{2(k-\ell)}$. Furthermore, assume that $\mathcal{H}$ is $\Delta$-extremal, namely, there is a set $B \subseteq V(\mathcal{H})$, such that $|B| = \lfloor \frac{(2k-2\ell-1)n}{2(k-\ell)} \rfloor$ and $e(B) \leq \Delta n^k$. Let $A = V \setminus B$. Then $|A| = \lceil \frac{n}{2(k-\ell)} \rceil$.

Let us give an outline of our proof first. We denote by $A'$ and $B'$ the sets of “typical” vertices of $A$ and $B$, respectively. Let $V_0 = V \setminus (A' \cup B')$. It is not hard to see that $A' \approx A$, $B' \approx B$, and thus $V_0 \approx \emptyset$. In the ideal case when $V_0 = \emptyset$ and $|B'| = (2k-2\ell-1)|A'|$, we assign a cyclic order to the vertices of $A'$, construct $|A'|$ copies of $\mathcal{Y}_{k,\ell}$ such that each copy contains one vertex of $A'$ and $2k-\ell-1$ vertices of $B'$, and any two consecutive copies of $\mathcal{Y}_{k,\ell}$ share exactly $\ell$ vertices of $B'$. This gives rise to the desired Hamilton $\ell$-cycle of $\mathcal{H}$. In the general case, we first construct an $\ell$-path $Q$ with ends $L_0$ and $L_1$ such that $V_0 \subseteq V(Q)$ and $|B_0| = (2k-2\ell-1)|A_1| + \ell$, where $A_1 = A' \setminus V(Q)$.
and $B_1 = (B \setminus V(Q)) \cup L_0 \cup L_1$. Next we complete the Hamilton $\ell$-cycle by constructing an $\ell$-path on $A_1 \cup B_1$ with ends $L_0$ and $L_1$.

For the convenience of later calculations, we let $\epsilon_0 = 2k!e\Delta \ll 1$ and claim that $e(B) \leq \epsilon_0(B_k)$. Indeed, since $2(k - \ell) - 1 \geq k$, we have

$$1/\epsilon \leq \left(1 - \frac{1}{2(k - \ell)}\right)^{2(k-\ell)-1} \leq \left(1 - \frac{1}{2(k - \ell)}\right)^k.$$ 

Thus we get

$$e(B) \leq \frac{\epsilon_0}{2k!} n^{k} \leq \epsilon_0 \left(1 - \frac{1}{2(k - \ell)}\right)^k n^{k} \leq \epsilon_0 \frac{[B]}{k}. \tag{3.1}$$

In general, given two disjoint vertex sets $X$ and $Y$ and two integers $i, j \geq 0$, a set $S \subseteq X \cup Y$ is called an $X^iY^j$-set if $|S \cap X| = i$ and $|S \cap Y| = j$. When $X, Y$ are two disjoint subsets of $V(H)$ and $i + j = k$, we denote by $H(X^iY^j)$ the family of all edges of $H$ that are $X^iY^j$-sets, and let $e_H(X^iY^j) = |H(X^iY^j)|$ (the subscript may be omitted if it is clear from the context). We use $e_H(X^iY^j)$ to denote the number of non-edges among $X^iY^j$-sets. Given a set $L \subseteq X \cup Y$ with $|L \cap X| = l_1 \leq i$ and $|L \cap Y| = l_2 \leq k - i$, we define $\deg(L, X^iY^j)$ as the number of edges in $H(X^iY^j)$ that contain $L$, and $\deg(L, X^iY^j) = \binom{|X|}{l_1} \binom{|Y|}{l_2} \deg(L, X^iY^j)$. Our earlier notation $\deg(S, R)$ may be viewed as $\deg(S, S^{[S]}(R \setminus S)^{k-|[S]|})$.

3.1. **Classification of vertices.** Let $\epsilon_1 = \epsilon_0^{1/3}$ and $\epsilon_2 = 2\epsilon_0$. Assume that the partition $V(H) = A \cup B$ satisfies that $|B| = \frac{2k - 2|\ell - 1)|n}{2(k-\ell)}$ and (3.1). In addition, assume that $e(B)$ is the smallest among all such partitions. We now define

$$A' := \left\{ v \in V : \deg(v, B) \geq (1 - \epsilon_1) \binom{|B|}{k - 1} \right\},$$

$$B' := \left\{ v \in V : \deg(v, B) \leq \epsilon_1 \binom{|B|}{k - 1} \right\},$$

$$V_0 := V \setminus (A' \cup B').$$

**Claim 3.1.** $A \cap B' \neq \emptyset$ implies that $B \subseteq B'$, and $B \cap A' \neq \emptyset$ implies that $A \subseteq A'$.

**Proof.** First, assume that $A \cap B' \neq \emptyset$. Then there is some $u \in A$ such that $\deg(u, B) \leq \epsilon_1 \binom{|B|}{k - 1}$. If there exists some $v \in B \setminus B'$, namely, $\deg(v, B) > \epsilon_1 \binom{|B|}{k - 1}$, then we can switch $u$ and $v$ and form a new partition $A'' \cup B''$ such that $|B''| = |B|$ and $e(B'') < e(B)$, which contradicts the minimality of $e(B)$.

Second, assume that $B \cap A' \neq \emptyset$. Then some $u \in B$ satisfies that $\deg(u, B) \geq (1 - \epsilon_1) \binom{|B|}{k - 1}$. Similarly, by the minimality of $e(B)$, we get that for any vertex $v \in A$, $\deg(v, B) \geq (1 - \epsilon_1) \binom{|B|}{k - 1}$, which implies that $A \subseteq A'$.

**Claim 3.2.** $|\{A \setminus A'|, |B \setminus B'|, |A' \setminus A|, |B' \setminus B\}| \leq \epsilon_2 |B|$ and $|V_0| \leq 2\epsilon_2 |B|$.

**Proof.** First assume that $|B \setminus B'| > \epsilon_2 |B|$. By the definition of $B'$, we get that

$$e(B) > \frac{1}{k} \epsilon_1 \binom{|B|}{k - 1} \cdot \epsilon_2 |B| > 2\epsilon_0 \binom{|B|}{k},$$

which contradicts (3.1).

Second, assume that $|A \setminus A'| > \epsilon_2 |B|$. Then by the definition of $A'$, for any vertex $v \notin A'$, we have that $\deg(v, B) > \epsilon_1 \binom{|B|}{k - 1}$. So we get

$$\pi(AB^{k-1}) > \epsilon_2 |B| \cdot \epsilon_1 \binom{|B|}{k - 1} = 2\epsilon_0 |B| \binom{|B|}{k - 1}.$$
Together with (3.1), this implies that
\[ \sum_{S \in \binom{[n]}{k-1}} \deg(S) = k\bar{\pi}(B) + \pi(AB^{k-1}) \]
\[ > k(1 - \epsilon_0)\frac{|B|}{k} + 2\epsilon_0|B|\frac{|B|}{k-1} \]
\[ = ((1 - \epsilon_0)(|B| - k + 1) + 2\epsilon_0|B|)\frac{|B|}{k - 1} > |B|\frac{|B|}{k - 1}. \]
where the last inequality holds because \( n \) is large enough. By the pigeonhole principle, there exists a set \( S \in \binom{[n]}{k-1} \), such that \( \overline{\deg}(S) > |B| = \left( \frac{2k - 2^{(k-1)}}{2^{(k-1)}} \right) \), contradicting (1.1).

Consequently,
\[ |A' \setminus A| = |A' \cap B| \leq |B \setminus B'| \leq \epsilon_2|B|, \]
\[ |B' \setminus B| = |A \cap B' \setminus B| \leq |A \setminus A'| \leq \epsilon_2|B|, \]
\[ |V_0| = |A \setminus A'| + |B \setminus B'| \leq \epsilon_2|B| + \epsilon_2|B| = 2\epsilon_2|B|. \]
\[ □ \]

3.2. Classification of \( \ell \)-sets in \( B' \). In order to construct our Hamilton \( \ell \)-cycle, we need to connect two \( \ell \)-paths. To make this possible, we want the ends of our \( \ell \)-paths to be \( \ell \)-sets in \( B' \) that have high degree in \( \mathcal{H}[A'B^{\ell-1}] \). Formally, we call an \( \ell \)-set \( L \subset V \) typical if \( \deg(L, B) \leq \epsilon_1\left( \frac{|B|}{k - 1} \right) \), otherwise atypical. We prove several properties related to typical \( \ell \)-sets in this subsection.

**Claim 3.3.** The number of atypical \( \ell \)-sets in \( B \) is at most \( \epsilon_2\left( \frac{|B|}{\ell} \right) \).

**Proof.** Let \( m \) be the number of atypical \( \ell \)-sets in \( B \). By (3.1), we have
\[ \frac{m\epsilon_1\left( \frac{|B|}{k - 1} \right)}{\ell} \leq \epsilon(B) \leq \epsilon_0\left( \frac{|B|}{k} \right), \]
which gives that
\[ m \leq \frac{\epsilon_0\left( \frac{\ell}{k} \right)}{\epsilon_1\left( \frac{k}{k - 1} \right)} \left( \frac{|B|}{\ell} \right) < \epsilon_2\left( \frac{|B|}{\ell} \right). \]
\[ □ \]

**Claim 3.4.** Every typical \( \ell \)-set \( L \subset B' \) satisfies \( \overline{\deg}(L, A'B^{\ell-1}) \leq 4\epsilon_1\left( \frac{|B'| - \ell}{k - \ell - 1} \right)\left| A' \right|. \)

**Proof.** Fix a typical \( \ell \)-set \( L \subset B' \), consider the following sum,
\[ \sum_{L \subset D \subset B', |D| = k-1} \deg(D) = \sum_{L \subset D \subset B', |D| = k-1} (\deg(D, A') + \deg(D, B') + \deg(D, V_0)). \]
By (1.1), the left hand side is at least \( (\frac{|B'| - \ell}{k - \ell - 1})\left| A' \right|. \)
On the other hand,
\[ \sum_{L \subset D \subset B', |D| = k-1} (\deg(D, B') + \deg(D, V_0)) \leq (k - \ell)\deg(L, B') + \left( \frac{|B'| - \ell}{k - \ell - 1} \right)|V_0|. \]
Since \( L \) is typical and \( |B' \setminus B| \leq \epsilon_2|B| \) (Claim 3.2), we have
\[ \deg(L, B') \leq \deg(L, B) + |B' \setminus B|\left( \frac{|B'| - 1}{k - \ell - 1} \right) \]
\[ \leq \epsilon_1\left( \frac{|B|}{k - \ell} \right) + \epsilon_2|B|\left( \frac{|B'| - 1}{k - \ell - 1} \right). \]
Since \( \epsilon_2 \ll \epsilon_1 \) and \( ||B| - |B'|| \leq \epsilon_2|B| \) (Claim 3.2), it follows that
\[ (k - \ell)\deg(L, B') \leq \epsilon_1|B|\left( \frac{|B| - 1}{k - \ell - 1} \right) + (k - \ell)\epsilon_2|B|\left( \frac{|B'| - 1}{k - \ell - 1} \right) \leq 2\epsilon_1|B|\left( \frac{|B'| - \ell}{k - \ell - 1} \right). \]
Putting these together and using Claim 3.2, we obtain that
\[
\sum_{L \subseteq D \subseteq B', |D| = k-1} \deg(D, A') \geq \binom{|B'| - \ell}{k - \ell - 1} (|A| - |V_0|) - 2\epsilon_1 |B| \binom{|B'| - \ell}{k - \ell - 1}
\geq \binom{|B'| - \ell}{k - \ell - 1} (|A'| - 3\epsilon_2 |B| - 2\epsilon_1 |B'|).
\]

Note that \( \deg(L, A'B'^{k-1}) = \sum_{L \subseteq D \subseteq B', |D| = k-1} \deg(D, A') \). Since \(|B| \leq (2k - 2\ell - 1) |A| \leq (2k - 2\ell) |A'| \), we finally derive that
\[
\deg(L, A'B'^{k-1}) \geq \binom{|B'| - \ell}{k - \ell - 1} (1 - (2k - 2\ell)(3\epsilon_2 + 2\epsilon_1)) |A'| \geq (1 - 4k\epsilon_1) \binom{|B'| - \ell}{k - \ell - 1} |A'|.
\]
as desired. \( \square \)

We next show that we can connect any two disjoint typical \( \ell \)-sets of \( B' \) with an \( \ell \)-path of length two while avoiding any given \( \frac{n}{4(k-\ell)} \) vertices of \( V \).

**Claim 3.5.** Given two disjoint typical \( \ell \)-sets \( L_1, L_2 \) in \( B' \) and a vertex set \( U \subseteq V \) with \(|U| \leq \frac{n}{4(k-\ell)} \), there exist a vertex \( a \in A' \setminus U \) and a \((2k - 3\ell - 1)-set \( C \subseteq B' \setminus U \) such that \( L_1 \cup L_2 \cup \{a\} \cup C \) spans an \( \ell \)-path (of length two) ended at \( L_1, L_2 \).

**Proof.** Fix two disjoint typical \( \ell \)-sets \( L_1, L_2 \) in \( B' \). Using Claim 3.2, we obtain that \(|U| \leq \frac{n}{4(k-\ell)} \leq \frac{|A|}{2} < \frac{3}{4} |A'| \) and
\[
\frac{n}{4(k-\ell)} \leq \frac{|B| + 1}{2(2k - 2\ell - 1)} < \frac{1 + 2\epsilon_2 |B'|}{2k} < \frac{|B'|}{k}.
\]
Thus \(|A' \setminus U| > \frac{|A'|}{3} \) and \(|B' \setminus U| > \frac{k-1}{k} |B'| \). Consider a \((k-\ell)\)-graph \( \mathcal{G} \) on \((A' \cup B') \setminus U \) such that an \( A'B'^{k-\ell-1} \)-set \( T \) is an edge of \( \mathcal{G} \) if and only if \( T \cap U = \emptyset \) and \( T \) is a common neighbor of \( L_1 \) and \( L_2 \) in \( \mathcal{H} \). By Claim 3.4, we have
\[
\tau(\mathcal{G}) \leq 2 \cdot 4k\epsilon_1 \left( \binom{|B'| - \ell}{k - \ell - 1} |A'| \right) < 8k\epsilon_1 \left( \frac{k}{k-1} |B'| |U| \right) \cdot \frac{3 |A'| |U|}{k-1}.
\]
Consequently \( e(\mathcal{G}) > \frac{1}{2} \left( \binom{|B'| |U|}{k-\ell-1} |A'| |U| \right) \). Hence there exists a vertex \( a \in A' \setminus U \) such that \( \deg_{\mathcal{G}}(a) > \frac{1}{2} \left( \binom{|B'| |U|}{k-\ell-1} \right) \). By Fact 2.9, the link graph of \( a \) contains a copy of \( \mathcal{Y}_{k-\ell-1,k-\ell-1} \) (two edges of the link graph sharing \( \ell - 1 \) vertices). In other words, there exists a \((2k - 3\ell - 1)\)-set \( C \subseteq B' \setminus U \) such that \( C \cup \{a\} \) contains two edges of \( \mathcal{G} \) sharing \( \ell \) vertices. Together with \( L_1, L_2 \), this gives rise to the desired \( \ell \)-path (in \( \mathcal{H} \)) of length two ended at \( L_1, L_2 \). \( \square \)

The following claim shows that we can always extend a typical \( \ell \)-set to an edge of \( \mathcal{H} \) by adding one vertex from \( A' \) and \( k - \ell - 1 \) vertices from \( B' \) such that every \( \ell \) new vertices form a typical \( \ell \)-set. This can be done even when at most \( \frac{n}{4(k-\ell)} \) vertices of \( V \) are not available.

**Claim 3.6.** Given a typical \( \ell \)-set \( L \subseteq B' \) and a set \( U \subseteq V \) with \(|U| \leq \frac{n}{4(k-\ell)} \), there exists an \( A'B'^{k-\ell-1} \)-set \( C \subseteq V \setminus U \) such that \( L \cup C \) is an edge of \( \mathcal{H} \) and every \( \ell \)-subset of \( C \cap B' \) is typical.

**Proof.** First, since \( L \) is typical in \( B' \), by Claim 3.4, \( \deg(L, A'B'^{k-1}) \leq 4k\epsilon_1 \left( \binom{|B'| - \ell}{k-\ell-1} |A'| \right) \). Second, note that a vertex in \( A' \) is contained in \( \left( \binom{|B'|}{k-\ell-1} \right) A'B'^{k-\ell-1} \)-sets, while a vertex in \( B' \) is contained
Claim 3.3, there are at most $\ell$ lower bound for $i < 2$ degree conditions (1.1). Suppose that Claim 3.7. A $B^{k-\ell-1}$-sets intersect $U$. Finally, by Claim 3.3, the number of atypical $\ell$-sets in $B$ is at most $c_2$ for each $B^{k-\ell-1}$-sets in $B$. Let $A' \subseteq B$, at most $2k^2c_2$ and $|B'| \approx 2k^{-2\ell-1}n$. We thus derive that at most

$$|U| = \binom{|B'|}{k-\ell-1} \leq \frac{n}{4(k-\ell)} \binom{|B'|}{k-\ell-1}$$

$A'B^{k-\ell-1}$-sets intersect $U$. Finally, by Claim 3.3, the number of atypical $\ell$-sets in $B$ is at most $c_2$ for each $B^{k-\ell-1}$-sets in $B$. Let $A' \subseteq B$, at most $2k^2c_2$ and $|B'| \approx 2k^{-2\ell-1}n$. We thus derive that at most $c_2 \left( \binom{|B|}{\ell} + |B' \setminus B| \binom{|B'| - 1}{\ell - 1} \leq 2c_2 \binom{|B'|}{\ell} + c_2 |B| \binom{|B'| - 1}{\ell - 1} \right) < 3\ell c_2 \binom{|B'|}{\ell}.$

Hence at most $3\ell c_2 \binom{|B'|}{\ell} |A'| \binom{|B'| - 1}{\ell - 1} A'B^{k-\ell-1}$-sets contain an atypical $\ell$-set. In summary, at most

$$4k c_1 \left( \binom{|B'| - \ell}{k - \ell - 1} |A'| + \frac{n}{4(k-\ell)} \binom{|B'|}{k-\ell-1} \right) + 3\ell c_2 \binom{|B'|}{\ell} \binom{|B'| - \ell}{k - 2\ell - 1} |A'|$$

$A'B^{k-\ell-1}$-sets fail some of the desired properties. Since $\epsilon_1, \epsilon_2 < 1$ and $|A'| \approx \frac{n}{2(k-\ell)}$, the desired $A'B^{k-\ell-1}$-set always exists.

3.3. Building a short path $Q$. The following claim is the only place where we used the exact codegree condition (1.1).

Claim 3.7. Suppose that $|A \cap B'| = q > 0$. Then there exists a family $P_1$ of vertex-disjoint 2$q$ edges in $B'$, each of which contains two disjoint typical $\ell$-sets.

Proof. Let $|A \cap B'| = q > 0$. Since $A \cap B' \neq \emptyset$, by Claim 3.1, we have $B \subseteq B'$, and consequently $|B'| = \frac{2k-2\ell-1}{2(k-\ell)} n + q$. By Claim 3.2, we have $q \leq |A \setminus A'| \leq c_2 |B|.$

Let $B$ denote the family of the edges in $B'$ that contain two disjoint typical $\ell$-sets. We derive a lower bound for $|B|$ as follows. We first pick a $(k-1)$-subset of $B$ (recall that $B \subseteq B'$) that contains no atypical $\ell$-subset. Since $2\ell \leq k - 1$, such a $(k-1)$-set contains two disjoint typical $\ell$-sets. By Claim 3.3, there are at most $c_2 \binom{|B'|}{\ell}$ atypical $\ell$-sets in $B \cap B' = B$ and in turn, there are at most $c_2 \binom{|B'|}{\ell} \binom{|B|-\ell}{k-\ell-1} \binom{k-1}{\ell} |B'|$ atypical $\ell$-subsets of $B$ that contain an atypical $\ell$-subset. Thus there are at least

$$\left( \binom{|B|}{k-1} - c_2 \frac{|B|}{\ell} \right) \binom{|B|-\ell}{k-\ell-1} = \left( 1 - \frac{k-1}{\ell} c_2 \right) \frac{|B|}{k-1}$$

$(k-1)$-subsets of $B$ that contain no atypical $\ell$-subset. After picking such a $(k-1)$-set $S \subseteq B$, we find a neighbor of $S$ by the codegree condition. Since $|B'| = \frac{2k-2\ell-1}{2(k-\ell)} n + q$, by (1.1), we have $\deg(S, B') \geq q$. We thus derive that

$$|B| \geq \left( 1 - \frac{k-1}{\ell} c_2 \right) \frac{|B'|}{k-1} \frac{q}{k},$$

in which we divide by $k$ because every edge of $B$ is counted at most $k$ times.

We claim that $B$ contains $2q$ disjoint edges. Suppose instead, a maximum matching in $B$ has $i < 2q$ edges. By the definition of $B$, for any vertex $b \in B'$, we have

$$\deg(b, B') \leq \deg(b, B) + |B' \setminus B| \frac{|B'| - 1}{k-2} \leq c_1 \frac{|B|}{k-1} + c_2 |B| \frac{|B'| - 1}{k-2} < 2c_1 \frac{|B|}{k-1}. \quad (3.2)$$
Thus at most $2qk \cdot 2c_1 \binom{|B|}{k-1}$ edges of $B'$ intersect the $i$ edges in the matching. Hence, the number of edges of $B$ that are disjoint from these $i$ edges is at least
\[
\frac{q}{k} \left( 1 - \left( \frac{k-1}{\ell} \right) \epsilon_2 \right) \binom{|B|}{k-1} - 4k \epsilon_1 \frac{|B|}{k-1} \geq \left( \frac{1}{k} - (4k + 1) \epsilon_1 \right) q \left( \frac{|B|}{k-1} \right) > 0,
\]
as $\epsilon_2 \ll \epsilon_1 \ll 1$. We may thus obtain a matching of size $i+1$, a contradiction. \hfill \Box

**Claim 3.8.** There exists a non-empty $\ell$-path $Q$ in $H$ with the following properties:

- $V_0 \subseteq V(Q)$,
- $|V(Q)| \leq 10k \epsilon_2 |B|$,
- two ends $L_0, L_1$ of $Q$ are typical $\ell$-sets in $B'$,
- $|B_1| = (2k - 2\ell - 1)|A_1| + \ell$, where $A_1 = A' \setminus V(Q)$ and $B_1 = (B' \setminus V(Q)) \cup L_0 \cup L_1$.

**Proof.** We split into two cases here.

**Case 1.** $A \cap B' \neq \emptyset$.

By Claim 3.1, $A \cap B' \neq \emptyset$ implies that $B \subseteq B'$. Let $q = |A \cap B'|$. We first apply Claim 3.7 and find a family $P_1$ of vertex-disjoint $2q$ edges in $B'$. Next we associate each vertex of $V_0$ with $2k - \ell - 1$ vertices of $B$ (so in $B'$) forming an $\ell$-path of length two such that these $|V_0|$ paths are pairwise vertex-disjoint, and also vertex-disjoint from the paths in $P_1$, and all these paths have typical ends. To see it, let $V_0 = \{x_1, \ldots, x_{|V_0|}\}$. Suppose that we have found such $\ell$-paths for $x_1, \ldots, x_{i-1}$ with $i \leq |V_0|$. Since $B \subseteq B'$, it follows that $A \setminus A' = (A \cap B') \cup V_0$. Hence $|V_0| + q = |A \setminus A'| \leq \epsilon_2 |B|$ by Claim 3.2. Therefore
\[
(2k - \ell - 1)(i - 1) + |V(P_1)| < 2k|V_0| + 2kq \leq 2k \epsilon_2 |B|
\]
and consequently at most $2k \epsilon_2 |B| \binom{|B| - 1}{k - 2} < 2k^2 \epsilon_2 \binom{|B|}{k - 1} (k - 1)$-sets of $B$ intersect the existing paths (including $P_1$). By the definition of $V_0$, $\deg(x_i, B) > \epsilon_1 \binom{|B|}{k-1}$. Let $G_{x_i}$ be the $(k - 1)$-graph on $B$ such that $e \in G_{x_i}$, if

- $\{x_i\} \cup e \in E(H)$,
- $e$ does not contain any vertex from the existing paths,
- $e$ does not contain any atypical $\ell$-set.

By Claim 3.3, the number of $(k - 1)$-sets in $B$ containing at least one atypical $\ell$-set is at most $\epsilon_2 \binom{|B|}{k-1} \binom{|B| - \ell}{k - 1} = \epsilon_2 \binom{|B|}{k-1} \binom{|B|}{k-1}$. Thus, we have
\[
e(G_{x_i}) \geq \epsilon_1 \binom{|B|}{k-1} - 2k^2 \epsilon_2 \binom{|B|}{k-1} - \epsilon_2 \binom{|B| - \ell}{k - 1} \binom{|B|}{k - 1} \geq \epsilon_1 \left( \frac{|B|}{k - 1} \right) > \left( \frac{|B|}{k - 2} \right),
\]
because $\epsilon_2 \ll \epsilon_1$ and $|B|$ is sufficiently large. By Fact 2.9, $G_{x_i}$ contains a copy of $\gamma_{k-1, \ell-1}$, which gives the desired $\ell$-path of length two containing $x_i$.

Denote by $P_2$ the family of $\ell$-paths we obtained so far. Now we need to connect paths of $P_2$ together to a single $\ell$-path. For this purpose, we apply Claim 3.5 repeatedly to connect the ends of two $\ell$-paths while avoiding previously used vertices. This is possible because $|V(P_2)| = (2k - \ell)|V_0| + 2kq$ and $2(k - 3\ell)(|V_0| + 2q - 1)$ vertices are needed to connect all the paths in $P_2$; the set $U$ (when we apply Claim 3.5) thus satisfies
\[
|U| \leq (4k - 4\ell)|V_0| + (6k - 6\ell)q - 2k + 3\ell \leq 6(k - \ell) \epsilon_2 |B| - 2k + 3\ell.
\]
Let $P$ denote the resulting $\ell$-path. We have $|V(P) \cap A'| = |V_0| + 2q - 1$ and
\[
|V(P) \cap B'| = k \cdot 2q + (2k - \ell - 1)|V_0| + (2k - 3\ell - 1)(|V_0| + 2q - 1)
= 2(2k - 2\ell - 1)|V_0| + 2(3k - 3\ell - 1)q - (2k - 3\ell - 1).
\]
Let $s = (2k - 2\ell - 1)|A' \setminus V(\mathcal{P})| - |B' \setminus V(\mathcal{P})|$. We have

$$s = (2k - 2\ell - 1)(|A'| - |V_0| - 2q + 1) - |B'| + 2(2k - 2\ell - 1)|V_0| + 2(3k - 3\ell - 1)q - (2k - 3\ell - 1)$$

$$= (2k - 2\ell - 1)|A'| - |B'| + (2k - 2\ell - 1)|V_0| + (2k - 2\ell)q + \ell.$$ 

Since $|A'| + |B'| + |V_0| = n$, we have

$$s = (2k - 2\ell)(|A'| + |V_0| + q) - n + \ell. \quad (3.3)$$

Note that $|A'| + |V_0| + q = |A|$ and

$$(2k - 2\ell)|A| - n = \begin{cases} 0, & \text{if } \frac{n}{k-\ell} \text{ is even} \\ k - \ell, & \text{if } \frac{n}{k-\ell} \text{ is odd.} \end{cases} \quad (3.4)$$

Thus $s = \ell$ or $s = k$. If $s = k$, then we extend $\mathcal{P}$ to an $\ell$-path $Q$ by applying Claim 3.6, otherwise let $Q = \mathcal{P}$. Then

$$|V(Q)| \leq |V(\mathcal{P})| + (k - \ell) \leq 6k\epsilon_2|B|,$$

and $Q$ has two typical ends $L_0, L_1 \subset B'$. We claim that

$$(2k - 2\ell - 1)|A' \setminus V(Q)| - |B' \setminus V(Q)| = \ell. \quad (3.5)$$

Indeed, when $s = \ell$, this is obvious; when $s = k$, $V(Q) \setminus V(\mathcal{P})$ contains one vertex of $A'$ and $k - \ell - 1$ vertices of $B'$ and thus

$$(2k - 2\ell - 1)|A' \setminus V(Q)| - |B' \setminus V(Q)| = s - (2k - 2\ell - 1) + (k - \ell - 1) = \ell.$$ 

Let $A_1 = A' \setminus V(Q)$ and $B_1 = (B' \setminus V(Q)) \cup L_0 \cup L_1$. We derive that $|B_1| = (2k - 2\ell - 1)|A_1| + \ell$ from (3.5).

**Case 2.** $A \cap B' = \emptyset$.

Note that $A \cap B' = \emptyset$ means that $B' \subseteq B$. Then we have

$$|A'| + |V_0| = |V \setminus B'| = |A| + |B \setminus B'|. \quad (3.6)$$

If $V_0 \neq \emptyset$, we handle this case similarly as in Case 1 except that we do not need to construct $\mathcal{P}_1$. By Claim 3.2, $|B \setminus B'| \leq \epsilon_2|B|$ and thus for any vertex $x \in V_0$,

$$\deg(x, B') \geq \deg(x, B) - |B \setminus B'| \cdot \left(\frac{|B|-1}{k-2}\right)$$

$$\geq \epsilon_1 \left(\frac{|B|}{k-1}\right) - (k-1)\epsilon_2 \left(\frac{|B|}{k-1}\right) > \frac{\epsilon_1}{2} \left(\frac{|B|}{k-1}\right). \quad (3.7)$$

As in Case 1, we let $V_0 = \{x_1, \ldots, x_{|V_0|}\}$ and cover them with vertex-disjoint $\ell$-paths of length two. Indeed, for each $i \leq |V_0|$, we construct $G_x$ as before and show that $e(G_x) \geq \frac{\epsilon_1}{4}(|B'|)$. We then apply Fact 2.9 to $G_x$, obtaining a copy of $Y_{k-1, \ell-1}$, which gives an $\ell$-path of length two containing $x_i$.

As in Case 1, we connect these paths to a single $\ell$-path $\mathcal{P}$ by applying Claim 3.5 repeatedly. Then $|V(\mathcal{P})| = (2k - \ell)|V_0| + (2k - 3\ell)(|V_0| - 1)$. Define $s$ as in Case 1. Thus (3.3) holds with $q = 0$. Applying (3.6) and (3.4), we derive that

$$s = 2(k - \ell)(|A| + |B \setminus B'|) - n + \ell = \begin{cases} \ell + 2(k - \ell)|B \setminus B'|, & \text{if } \frac{n}{k-\ell} \text{ is even} \\ k + 2(k - \ell)|B \setminus B'|, & \text{if } \frac{n}{k-\ell} \text{ is odd,} \end{cases} \quad (3.8)$$

which implies that $s \equiv \ell \mod (k - \ell)$. We extend $\mathcal{P}$ to an $\ell$-path $\mathcal{Q}$ by applying Claim 3.6 $\frac{s-\ell}{k-\ell}$ times. Then

$$|V(\mathcal{Q})| = |V(\mathcal{P})| + s - \ell \leq (4k - 4\ell)|V_0| - 2k + 3\ell + k - \ell + 2(k - \ell)|B \setminus B'| \leq 10k\epsilon_2|B|.$$
by Claim 3.2. Note that \( Q \) has two typical ends \( L_0, L_1 \subset B' \). Since \( V(Q) \setminus V(P) \) contains \( \frac{s - \ell}{k - \ell} \) vertices of \( A' \) and \( \frac{s - \ell}{k - \ell} (k - \ell - 1) \) vertices of \( B' \), we have

\[
(2k - 2\ell - 1)|A' \setminus V(Q)| - |B' \setminus V(Q)| = s - \frac{s - \ell}{k - \ell}(2k - 2\ell - 1) + \frac{s - \ell}{k - \ell}(k - \ell - 1) = \ell.
\]

We define \( A_1 \) and \( B_1 \) in the same way and similarly we have \( |B_1| = (2k - 2\ell - 1)|A_1| + \ell \).

When \( V_0 = \emptyset \), we pick an arbitrary vertex \( v \in A' \) and form an \( \ell \)-path \( P \) of length two with typical ends such that \( v \) is in the intersection of the two edges. This is possible by the definition of \( A' \). Define \( s \) as in Case 1. It is easy to see that (3.8) still holds. We then extend \( P \) to \( Q \) by applying Claim 3.6 \( \frac{2k - 2\ell}{k - \ell} \) times. Then \( |V(Q)| = 2k - \ell + s - \ell \leq 2k\epsilon_2 |B| \) because of (3.8). The rest is the same as in the previous case.

\[ \square \]

**Claim 3.9.** The \( A_1, B_1 \) and \( L_0, L_1 \) defined in Claim 3.8 satisfy the following properties:

1. \(|B_1| \geq (1 - \epsilon_1)|B| \).
2. For any vertex \( v \in A_1 \), \( \overline{\deg}(v, B_1) < 3\epsilon_1 \left( \frac{|B_1|}{k - 1} \right) \).
3. For any vertex \( v \in B_1 \), \( \deg(v, A_1 B_1^{k-1}) < 3k\epsilon_1 \left( \frac{|B_1|}{k - 1} \right) \).
4. \( \overline{\deg}(L_0, A_1 B_1^{k-1}) \leq 5k\epsilon_1 \left( \frac{|B_1|}{k - 1} \right) \).}

**Proof.** Part (1): By Claim 3.2, we have \(|B_1 \setminus B| \leq |B' \setminus B| \leq \epsilon_2 |B| \). Furthermore,

\[
|B_1| \geq |B'| - |V(Q)| \geq |B| - \epsilon_2 |B| - 10k\epsilon_2 |B| \geq (1 - \epsilon_1)|B|.
\]

Part (2): For a vertex \( v \in A_1 \), since \( \overline{\deg}(v, B_1) \leq \epsilon_1 \left( \frac{|B_1|}{k - 1} \right) \), we have

\[
\overline{\deg}(v, B_1) \leq \overline{\deg}(v, B) + |B_1 \setminus B| \left( \frac{|B_1|}{k - 2} \right)
\]

\[
\leq \epsilon_1 \left( \frac{|B|}{k - 1} \right) + \epsilon_2 |B| \left( \frac{|B_1|}{k - 2} \right)
\]

\[
< \epsilon_1 \left( \frac{|B|}{k - 1} \right) + \epsilon_1 \left( \frac{|B_1|}{k - 1} \right) < 3\epsilon_1 \left( \frac{|B_1|}{k - 1} \right),
\]

where the last inequality follows from Part (1).

Part (3): Consider the sum \( \sum \deg(S \cup \{v\}) \) taken over all \( S \in \binom{B' \setminus \{v\}}{k - 2} \). Since \( \delta_{k-1}(H) \geq |A| \), we have \( \sum \deg(S \cup \{v\}) \geq \binom{|B| - 1}{k - 2} |A| \). On the other hand,

\[
\sum \deg(S \cup \{v\}) = \deg(v, A'B^{k-1}) + \deg(v, V_0 B^{k-1}) + (k - 1) \deg(v, B').
\]

By (3.2), \( \deg(v, B') \leq 2\epsilon_1 \left( \frac{|B_1|}{k - 1} \right) \). We thus derive that

\[
\deg(v, A'B^{k-1}) \geq \left( \frac{|B'| - 1}{k - 2} \right) |A| - \deg(v, V_0 B^{k-1}) - (k - 1) \deg(v, B')
\]

\[
\geq \left( \frac{|B'| - 1}{k - 2} \right) (|A' - \epsilon_2 |B|) - 2\epsilon_2 |B| \left( \frac{|B'| - 1}{k - 2} \right) - 2(k - 1)\epsilon_1 \left( \frac{|B|}{k - 1} \right)
\]

\[
\geq \left( \frac{|B'| - 1}{k - 2} \right) |A'| - 2k\epsilon_1 \left( \frac{|B|}{k - 1} \right).
\]

Thus, by Part (1), we have

\[
\overline{\deg}(v, A_1 B_1^{k-1}) \leq \overline{\deg}(v, A'B^{k-1}) \leq 2k\epsilon_1 \left( \frac{|B|}{k - 1} \right) \leq 3k\epsilon_1 \left( \frac{|B_1|}{k - 1} \right).
\]
Part (4): By Claim 3.4, for any typical $L \subseteq B'$, we have $\overline{\deg}(L, A'B^{k-1}) \leq 4k\epsilon_1(\frac{|B'|}{k-\ell}) |A'|$. Thus,
\[
\overline{\deg}(L_0, A_1 B_1^{k-1}) \leq \overline{\deg}(L_0, A'B^{k-1}) \leq 4k\epsilon_1(\frac{|B'|}{k-\ell}) |A'| \leq 5k\epsilon_1(\frac{|B_1|}{k-\ell}),
\]
where the last inequality holds because $|B'| \leq |B_1| + |V(G)| \leq (1 + \epsilon_1)|B_1|$. The same holds for $L_1$.

3.4. Completing the Hamilton cycle. We finally complete the proof of Theorem 1.5 by applying the following lemma with $X = A_1$, $Y = B_1$, $\rho = 5k\epsilon_1$, and $L_0, L_1$.

**Lemma 3.10.** Fix $1 \leq \ell < k/2$. Let $0 < \rho \ll 1$ and $n$ be sufficiently large. Suppose that $\mathcal{H}$ is a $k$-graph with a partition $V(\mathcal{H}) = X \cup Y$ and the following properties:

- $|Y| = (2k - 2\ell - 1)|X| + \ell$,
- for every vertex $v \in X$, $\overline{\deg}(v, Y) \leq \rho\binom{|Y|}{k-\ell}$ and for every vertex $v \in Y$, $\overline{\deg}(v, XY^{k-1}) \leq \rho\binom{|Y|}{k-\ell}$,
- there are two disjoint $\ell$-sets $L_0, L_1 \subset Y$ such that
\[
\overline{\deg}(L_0, XY^{k-1}), \overline{\deg}(L_1, XY^{k-1}) \leq \rho\binom{|Y|}{k-\ell}. \tag{3.9}
\]

Then $\mathcal{H}$ contains a Hamilton $\ell$-path with $L_0$ and $L_1$ as ends.

In order to prove Lemma 3.10, we apply two results of Glebov, Person, and Weps [6]. Given $1 \leq \ell \leq k - 1$ and $0 \leq \rho \leq 1$, an ordered set $(x_1, \ldots, x_t)$ is $\rho$-typical in a $k$-graph $\mathcal{G}$ if for every $i \in [t]$
\[
\overline{\deg}_\mathcal{G}((x_1, \ldots, x_i)) \leq \rho^{-i}\binom{|V(\mathcal{G})| - i}{k - i}.
\]

It was shown in [6] that every $k$-graph $\mathcal{G}$ with very large minimum vertex degree contains a tight Hamilton cycle. The proof of [6, Theorem 2] actually shows that we can obtain a tight Hamilton cycle by extending any fixed tight path of constant length with two typical ends. This implies the following theorem that we will use.

**Theorem 3.11.** [6] Given $1 \leq \ell \leq k$ and $0 < \alpha \ll 1$, there exists an $m_0$ such that the following holds. Suppose that $\mathcal{G}$ is a $k$-graph on $V$ with $|V| = m \geq m_0$ and $\delta_1(\mathcal{G}) \geq (1 - \alpha)|V|^{k-1}$. Then given any two $(2\alpha k^{k-1})$-typical ordered $\ell$-sets $(x_1, \ldots, x_{\ell})$ and $(y_1, \ldots, y_{\ell})$, there exists a tight Hamilton path $P = x_1 x_{\ell-1} \cdots x_1 \cdots y_1 y_{\ell} \cdots y_1$ in $\mathcal{G}$.

We also use [6, Lemma 3], in which $V^{2k-2}$ denotes the set of all $(2k - 2)$-tuples $(v_1, \ldots, v_{2k-2})$ such that $v_i \in V$ ($v_i$’s are not necessarily distinct).

**Lemma 3.12.** [6] Let $\mathcal{G}$ be the $k$-graph given in Lemma 3.11. Suppose that $(x_1, \ldots, x_{2k-2})$ is selected uniformly at random from $V^{2k-2}$. Then the probability that all $x_i$’s are pairwise distinct and $(x_1, \ldots, x_{k-1}), (x_{2k-2} - x_{2k-2})$ are $(2\alpha k^{k-1})$-typical is at least $\frac{8}{11}$.

**Proof of Lemma 3.10.** In this proof we often write the union $A \cup B \cup \{x\}$ as $ABx$, where $A, B$ are sets and $x$ is an element.

Let $t = |X|$. Our goal is to write $X$ as $\{x_1, \ldots, x_t\}$ and partition $Y$ as $\{L_i, R_i, S_i, R'_i : i \in [t]\}$ with $|L_i| = \ell$, $|R_i| = |R'_i| = k - 2\ell$, and $|S_i| = \ell - 1$ such that
\[
L_i R_i S_i x_i, S_i x_{i+1} R'_i L_{i+1} \in E(\mathcal{H}) \tag{3.10}
\]
for all $i \in [t]$, where $L_{i+1} = L_0$. Consequently
\[
L_1 R_1 S_1 x_1 R'_1 L_2 R_2 S_2 x_2 R'_2 \cdots L_t R_t S_t x_t R'_t L_{t+1}
\]
is the desired Hamilton \(t\)-path of \(\mathcal{H}\).

Let \(\mathcal{G}\) be the \((k-1)\)-graph on \(Y\) whose edges are all \((k-1)\)-sets \(S \subseteq Y\) such that \(\text{deg}_{\mathcal{H}}(S,X) > (1 - \sqrt{\rho})t\). The following is an outline of our proof. We first find a small subset \(Y_0 \subseteq Y\) with a partition \(\{L_i, R_i, S_i, R'_i : i \in [t_0]\}\) such that for every \(x \in X\), we have \(L_i R_i S_i x, S_i x R'_i L_{i+1} \in E(\mathcal{H})\) for every \(i \in [t_0]\). Next we apply Theorem 3.11 to \(G\) and obtain a tight Hamilton path, which, in particular, partitions \(Y\) into \(\{L_i, R_i, S_i, R'_i : t_0 < i \leq t\}\) such that \(L_i R_i S_i, S_i R'_i L_{i+1} \in E(\mathcal{G})\) for \(t_0 < i \leq t\). Finally we apply the Marriage Theorem to find a perfect matching between \(X\) and \([t]\) such that (3.10) holds for all matched \(x_i\) and \(i\).

We now give details of the proof. First we claim that
\[
\delta_1(\mathcal{G}) \geq (1 - 2\sqrt{\rho})(|Y| - 1) \quad (3.11)
\]
and consequently,
\[
\tau(\mathcal{G}) \leq 2\sqrt{\rho}\left(\frac{|Y|}{k-1}\right) \quad (3.12)
\]
Suppose instead, some vertex \(v \in Y\) satisfies \(\text{deg}_{\mathcal{G}}(v) > 2\sqrt{\rho}\left(\frac{|Y| - 1}{k-2}\right)\). Since every non-neighbor \(S'\) of \(v\) in \(\mathcal{G}\) satisfies \(\text{deg}_{\mathcal{H}}(S'v, X) \geq \sqrt{\rho}t\), we have \(\text{deg}_{\mathcal{H}}(v, Y^{k-1}) > 2\sqrt{\rho}\left(\frac{|Y| - 1}{k-2}\right)\). Since \(|Y| = (2k - 2k - 1) + \ell\), we have
\[
\text{deg}_{\mathcal{H}}(v, Y^{k-1}) > 2\rho \frac{|Y| - \ell}{2k - 2k - 1} \left(\frac{|Y| - 1}{k-2}\right) > \rho \left(\frac{|Y|}{k-1}\right) \left(\frac{|Y| - 1}{k-2}\right) = \rho \left(\frac{|Y|}{k-1}\right),
\]
contradicting our assumption (the second inequality holds because \(|Y|\) is sufficiently large).

Let \(Q\) be a \((2k - \ell - 1)\)-subset of \(Y\). We call \(Q\) good (otherwise bad) if every \((k - 1)\)-subset of \(Q\) is an edge of \(\mathcal{G}\) and every \(\ell\)-set \(L \subset Q\) satisfies
\[
\text{deg}_{\mathcal{G}}(L) \leq \rho^{1/4}\left(\frac{|Y| - \ell}{k - \ell - 1}\right) \quad (3.13)
\]
Furthermore, we say \(Q\) is suitable for a vertex \(x \in X\) if \(x \cup T \in E(\mathcal{H})\) for every \((k - 1)\)-set \(T \subset Q\). Note that if a \((2k - \ell - 1)\)-set is good, by the definition of \(\mathcal{G}\), it is suitable for at least \((1 - \rho^{1/5})\left(\frac{|Y|}{2k - \ell - 1}\right)\) vertices of \(X\). Let \(Y' = Y \setminus (L_0 \cup L_1)\).

Claim 3.13. For any \(x \in X\), at least \((1 - \rho^{1/5})\left(\frac{|Y|}{2k - \ell - 1}\right)\) \((2k - \ell - 1)\)-subsets of \(Y'\) are good and suitable for \(x\).

Proof. Since \(\rho + \rho^{1/2} + 3(2k - \ell - 1)\rho^{1/4} \leq \rho^{1/5}\), the claim follows from the following three assertions:
- At most \(2\ell\left(\frac{|Y| - 1}{2k - \ell - 2}\right) \leq \rho\left(\frac{|Y| - 1}{2k - \ell - 1}\right)\) \((2k - \ell - 1)\)-subsets of \(Y\) are not subsets of \(Y'\).
- Given \(x \in X\), at most \(\rho^{1/2}\left(\frac{|Y| - 1}{2k - \ell - 1}\right)\) \((2k - \ell - 1)\)-sets in \(Y\) are not suitable for \(x\).
- At most \(3\left(\frac{|Y| - 1}{2k - \ell - 1}\right)\rho^{1/4}\left(\frac{|Y|}{2k - \ell - 1}\right)\) \((2k - \ell - 1)\)-sets in \(Y\) are bad.

The first assertion holds because \(|Y' - Y'| = 2\ell\). The second assertion follows from the degree condition of \(\mathcal{H}\), namely, for any \(x \in X\), the number of \((2k - \ell - 1)\)-sets in \(Y\) that are not suitable for \(x\) is at most \(\rho\left(\frac{|Y|}{k - \ell - 1}\right)\left(\frac{|Y| - k - 1}{k - \ell - 1}\right) \leq \sqrt{\rho}\left(\frac{|Y|}{2k - \ell - 1}\right)\).

To see the third one, let \(m\) be the number of \(\ell\)-sets \(L \subseteq Y\) that fail (3.13). By (3.12),
\[
m \cdot \frac{\rho^{1/4}\left(\frac{|Y| - \ell}{k - \ell - 1}\right)}{\left(\frac{|Y|}{2k - \ell - 1}\right)} \leq \tau(\mathcal{G}) \leq 2\sqrt{\rho}\left(\frac{|Y|}{k - 1}\right),
\]
which implies that \(m \leq 2\rho^{1/4}\left(\frac{|Y|}{\ell}\right)\). Thus at most
\[
2\rho^{1/4}\left(\frac{|Y|}{\ell}\right) \cdot \left(\frac{|Y| - \ell}{2k - 2\ell - 1}\right)
\]
$(2k-\ell-1)$-subsets of $Y$ contain an $\ell$-set $L$ that fails (3.13). On the other hand, by (3.12), at most
\[ \tau(G) \binom{|Y| - k + 1}{k - \ell} \leq 2\sqrt{\rho} \binom{|Y|}{k - 1} \binom{|Y| - k + 1}{k - \ell} \]
$(2k-\ell-1)$-subsets of $Y$ contain a non-edge of $G$. Putting these together, the number of bad $(2k-\ell-1)$-sets in $Y$ is at most
\[ 2\rho^{1/4} \binom{|Y|}{\ell} \binom{|Y| - \ell}{2k - 2\ell - 1} + 2\sqrt{\rho} \binom{|Y|}{k - 1} \binom{|Y| - k + 1}{k - \ell} \leq 3(2k - \ell - 1)\rho^{1/4} \binom{|Y|}{2k - \ell - 1}, \]
as $\rho \ll 1$.

We will pick a family of disjoint good $(2k-\ell-1)$-sets in $Y'$ such that for any $x \in X$, many members of this family are suitable for $x$. To achieve this, we pick a family $F$ by selecting each good $(2k-\ell-1)$-sets of $Y'$ randomly and independently with probability $p = 6\sqrt{\rho}|Y|/(2k-\ell-1)$. Since there are at most $(2k-\ell-1)(2k-\ell-2)$ pairs of intersecting $(2k-\ell-1)$-sets in $Y'$, the expected number of intersecting pairs of $(2k-\ell-1)$-sets in $F$ is at most
\[ p^2 \binom{|Y|}{2k - \ell - 1} \binom{|Y| - 1}{2k - \ell - 2} = 36(2k - \ell - 1)^2 \rho |Y|. \]

By applying Chernoff’s bound on the first two properties and Markov’s bound on the last one below, we can find, with positive probability, a family $F$ of good $(2k-\ell-1)$-subsets of $Y'$ that satisfies
- $|F| \leq 2p(2k-\ell-1)^2 \leq 12\sqrt{\rho}|Y|$,
- for any vertex $x \in X$, because of Claim 3.13, at least
\[ \frac{p}{2}(1 - \rho^{1/5}) \binom{|Y|}{2k - \ell - 1} \geq 2\sqrt{\rho}|Y| \]
members of $F$ are suitable for $x$.
- the number of intersecting pairs of $(2k-\ell-1)$-sets in $F$ is at most $72(2k-\ell-1)^2 \rho |Y|$.

After deleting one $(2k-\ell-1)$-set from each of the intersecting pairs from $F$, we obtain a family $F' \subseteq F$ consisting of at most $12\sqrt{\rho}|Y|$ disjoint good $(2k-\ell-1)$-subsets of $Y'$ and for each $x \in X$, at least
\[ 2\sqrt{\rho}|Y| - 72(2k - \ell - 1)^2 \rho |Y| \geq \frac{3}{2} \sqrt{\rho}|Y| \]
members of $F'$ are suitable for $x$.

Denote $F'$ by $\{Q_2, Q_4, \ldots, Q_{2q}\}$ for some $q \leq 12\sqrt{\rho}|Y|$. We arbitrarily partition each $Q_{2i}$ into $L_{2i} \cup P_{2i} \cup L_{2i+1}$ such that $|L_{2i}| = |L_{2i+1}| = \ell$ and $|P_{2i}| = 2k - 3\ell - 1$. Since $Q_{2i}$ is good, both $L_{2i}$ and $L_{2i+1}$ satisfy (3.13). We claim that $L_0$ and $L_1$ satisfy (3.13) as well. Let us show this for $L_0$. By the definition of $G$, the number of $XY^{k-\ell-1}$-sets $T$ such that $T \cup L_0 \not\in E(H)$ is at least $\deg_G(L_0)\sqrt{\rho}t$. Using (3.9), we derive that $\deg_G(L_0)\sqrt{\rho}t \leq \rho^{(Y)}_{k-\ell-1}$. Since $|Y| \leq (2k-2\ell)t$, it follows that $\deg_G(L_0) \leq 2\sqrt{\rho}^{(Y)-1}_{k-\ell-1} \leq \rho^{1/4}^{(Y)-1}_{k-\ell-1}$.

Next we find disjoint $(2k-3\ell-1)$-sets $P_1, P_3, \ldots, P_{2q-1}$ from $Y' \setminus \bigcup_{i=1}^{q} Q_{2i}$ such that for $i \in [q]$, every $(k-\ell-1)$-subset of $P_{2i-1}$ is a common neighbor of $L_{2i-1}$ and $L_{2i}$ in $G$. Since $L_1, L_2, \ldots, L_{2q}$ all satisfy (3.13), at most
\[ 2 \cdot \rho^{1/4} \binom{|Y| - \ell}{k - \ell - 1} \binom{|Y| - k + \ell + 1}{k - 2\ell} \]
$(2k-3\ell-1)$-subsets of $Y$ contain a non-neighbor of $L_{2i-1}$ or $L_{2i}$. Since $q \leq 12\sqrt{\rho}|Y|$ and $\rho \ll 1$, we can greedily find desired $P_1, P_3, \ldots, P_{2q-1}$.
Let \( Y_1 = Y' \setminus \bigcup_{i=1}^q (P_{2i-1} \cup Q_{2i}) \) and \( G' = G[Y_1] \). Then \( |Y_1| = |Y'| - (2k - 2\ell - 1)2q \). Since \( \deg_{G'}(v) \leq \deg_G(v) \) for every \( v \in Y_1 \), we have, by (3.11),

\[
\delta_1(G') \geq \left( |Y| - \frac{1}{2} \right) - 2\sqrt{t} \left( |Y| - \frac{1}{2} \right) \geq (1 - 3\sqrt{t}) \left( |Y| - \frac{1}{2} \right).
\]

Let \( \alpha = 3\sqrt{t} \) and \( \rho_0 = (22\alpha)^{\frac{1}{4t}} \). We want to find two disjoint \( \rho_0 \)-typical ordered subsets \((x_1, \ldots, x_{k-\ell-1})\) and \((y_1, \ldots, y_{k-\ell-1})\) of \( Y_1 \) such that

\[
L_{2q+1} \cup \{x_1, \ldots, x_{k-\ell-1}\}, \ L_0 \cup \{y_1, \ldots, y_{k-\ell-1}\} \in E(G).
\]

To achieve this, we choose \((x_1, \ldots, x_{k-\ell-1}, y_1, \ldots, y_{k-\ell-1})\) from \( Y_1^{2k-2} \) uniformly at random. By Lemma 3.12, with probability at least \( \frac{k}{k} \), \((x_1, \ldots, x_{k-\ell-1})\) and \((y_1, \ldots, y_{k-\ell-1})\) are two disjoint ordered \( \rho_0 \)-typical \((k - \ell - 1)\)-sets. Since \( L_0 \) satisfies (3.13), at most \((k - \ell - 1)!q^{1/4}(\sqrt{k})^{1-\ell})\) ordered \((k - \ell - 1)\)-subsets of \( Y \) are not neighbors of \( L_0 \) (the same holds for \( L_{2q+1} \)). Thus (3.15) fails with probability at most \( 2(k - \ell - 1)!q^{1/4} \), provided that \( x_1, \ldots, x_{k-\ell-1}, y_1, \ldots, y_{k-\ell-1} \) are all distinct. Therefore the desired \((x_1, \ldots, x_{k-\ell-1})\) and \((y_1, \ldots, y_{k-\ell-1})\) exist.

Next we apply Theorem 3.11 to \( G' \) and obtain a tight Hamilton path

\[
P = x_{k-\ell-1}x_{k-\ell-2}\cdots x_1 \cdots \cdots y_1y_2 \cdots y_{k-\ell-1}.
\]

Following the order of \( P \), we partition \( Y_1 \) into

\[
R_{2q+1}, S_{2q+1}, R'_{2q+1}, L_{2q+2}, \ldots, L_t, R_t, S_t, R'_t
\]

such that \( |L_i| = \ell, |R_i| = |R'_i| = k - 2\ell \), and \( |S_i| = \ell - 1 \). Since \( P \) is a tight path in \( G \), we have

\[
L_i R_i S_i, S_i R'_i L_{i+1} \in E(G)
\]

for \( 2q + 2 \leq i \leq t - 1 \). Letting \( L_{i+1} = L_0 \), by (3.15), we also have (3.16) for \( i = 2q + 1 \) and \( i = t \).

We now arbitrarily partition \( P_i, 1 \leq i \leq 2q \) into \( R_i \cup S_i \cup R'_i \) such that \( |R_i| = |R'_i| = k - 2\ell \), and \( |S_i| = \ell - 1 \). By the choice of \( P_i \), (3.16) holds for \( 1 \leq i \leq 2q \).

Consider the bipartite graph \( \Gamma \) between \( X \) and \( Z := \{z_1, z_2, \ldots, z_t\} \) such that \( x \in X \) and \( z_i \in Z \) are adjacent if and only if \( L_i R_i S_i x S_i R'_i L_{i+1} \in E(H) \). For every \( i \leq t \), since (3.16) holds, we have \( \deg_{\Gamma}(z_i) \geq (1 - 2\sqrt{t}) \) by the definition of \( G' \). Let \( Z' = \{z_{2q+1}, \ldots, z_t\} \) and \( X_0 \) be the set of \( x \in X \) such that \( \deg_{\Gamma}(x, Z') \leq |Z'|/2 \). Then

\[
|X_0| \frac{|Z'|}{2} \leq \sum_{x \in X} \deg_{\Gamma}(x, Z') \leq 2\sqrt{t} \cdot |Z'|
\]

which implies that \( |X_0| \leq 4\sqrt{t} = 4\sqrt{t} \cdot \frac{|Y| - \ell}{2k - 2\ell - 1} \leq 4\sqrt{t} \cdot |Y| \) (note that \( 2k - 2\ell - 1 \geq k \geq 3 \)).

We now find a perfect matching between \( X \) and \( Z \) as follows.

Step 1: Each \( x \in X_0 \) is matched to some \( z_{2i}, i \in [q] \) such that the corresponding \( Q_{2i} \in F' \) is suitable for \( x \) (thus \( x \) and \( z_{2i} \) are adjacent in \( \Gamma \)) – this is possible because of (3.14) and \( |X_0| \leq 4\sqrt{t}|Y| \).

Step 2: Each of the unused \( z_i, i \in [2q] \) is matched to a vertex in \( X \setminus X_0 \) – this is possible because \( \deg_{\Gamma}(z_i) \geq (1 - 2\sqrt{t}) \geq |X_0| + 2q \).

Step 3: Let \( X' \) be the set of the remaining vertices in \( X \). Then \( |X'| = t - 2q = |Z'| \). Now consider the induced subgraph \( \Gamma' \) of \( \Gamma \) on \( X' \cup Z' \). Since \( \delta(\Gamma') \geq |X'|/2 \), the Marriage Theorem provides a perfect matching in \( \Gamma' \).

The perfect matching between \( X \) and \( Z \) gives rise to the desired Hamilton path of \( H \). \( \square \)
Let \( h_d^\ell(k, n) \) denote the minimum integer \( m \) such that every \( k \)-graph \( H \) on \( n \) vertices with minimum \( d \)-degree \( \delta_d(H) \geq m \) contains a Hamilton \( \ell \)-cycle (provided that \( k - \ell \) divides \( n \)). In this paper we determined \( h^\ell_{k-1}(k, n) \) for all \( \ell < k/2 \) and sufficiently large \( n \). Unfortunately our proof does not give \( h^\ell_{k-1}(k, n) \) for all \( k, \ell \) such that \( k - \ell \) does not divide \( k \) even though we believe that \( h^\ell_{k-1}(k, n) = \left\lceil \frac{n}{k-\ell} \right\rceil \). In fact, when \( k - \ell \) does not divide \( k \), if we can prove a path-cover lemma similar to Lemma 2.3, then we can follow the proof in [13] to solve the nonextremal case. When \( \ell \geq k/2 \), we cannot define \( Y_{k,2\ell} \) so the current proof of Lemma 2.3 fails. In addition, when \( \ell \geq k/2 \), the extremal case becomes complicated as well.

The situation is quite different when \( k - \ell \) divides \( k \). When \( k \) divides \( n \), one can easily construct a \( k \)-graph \( H \) such that \( \delta_{k-1}(H) \geq \frac{k}{2} - k \) and yet \( H \) contains no perfect matching and consequently no Hamilton \( \ell \)-cycle for any \( \ell \) such that \( k - \ell \) divides \( k \). A construction in [15] actually shows that \( h^\ell_{k-1}(k, n) \geq \frac{k}{2} - k \) whenever \( k - \ell \) divides \( k \), even when \( k \) does not divide \( n \). The exact value of \( h^\ell_d(k, n) \) is not known except for \( h^\ell_2(3, n) = \lfloor n/2 \rfloor \) given in [20]. In the forthcoming paper [8], the first author determines \( h^{k/2}_d(k, n) \) exactly for even \( k \) and any \( d \geq k/2 \).

Let \( t_d(n, F) \) denote the minimum integer \( m \) such that every \( k \)-graph \( H \) on \( n \) vertices with minimum \( d \)-degree \( \delta_d(H) \geq m \) contains a perfect \( F \)-tiling. One of the first results on hypergraph tiling was \( t_2(n, Y_{2,2}) = n/4 + o(n) \) given by Kühn and Osthus [14]. The exact value of \( t_2(n, Y_{3,2}) \) was determined recently by Czygrinow, DeBiasio, and Nagle [2]. We [10] determined \( t_3(n, Y_{3,2}) \) very recently. The key lemma in our proof, Lemma 2.8, shows that every \( k \)-graph \( H \) on \( n \) vertices with \( \delta_{k-1}(H) \geq (\frac{1}{2k-b} - o(1))n \) either contains an almost perfect \( Y_{k,b} \)-tiling or is in the extremal case. Naturally this raises a question: what is \( t_{k-1}(n, Y_{k,b}) \)? Mycroft [16] recently proved a general result on tiling \( k \)-partite \( k \)-graphs, which implies that \( t_{k-1}(n, Y_{k,b}) = \frac{n}{2k-b} + o(n) \). The lower bound comes from the following construction. Let \( H_0 \) be the \( k \)-graph on \( n \in (2k-b)\mathbb{N} \) vertices such that \( V(H_0) = A \cup B \) with \( \left| A \right| = \frac{n}{2k-b} - 1 \), and \( E(H_0) \) consists of all \( k \)-sets intersecting \( A \) and some \( k \)-subsets of \( B \) such that \( H_0[B] \) contains no copy of \( Y_{k,b} \). Thus, \( \delta_{k-1}(H_0) \geq \frac{1}{2k-b} - 1 \). Since every copy of \( Y_{k,b} \) contains at least one vertex in \( A \), there is no perfect \( Y_{k,b} \)-tiling in \( H_0 \). We believe that one can find a matching upper bound by the absorbing method (similar to the proof in [2]). In fact, since we already proved Lemma 2.8, it suffices to prove an absorbing lemma and the extremal case.

**References**

8. J. Han. Minimum degree conditions for \((k/2)\)-Hamilton cycles in \( k \)-graphs. *preprint*.
10. J. Han and Y. Zhao. Minimum vertex degree threshold for \((C_5^3)\)-tiling. *submitted*.


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