Proof of the $(n/2 - n/2 - n/2)$ Conjecture for large n

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Abstract

A conjecture of Loebl, also known as the $(n/2 - n/2 - n/2)$ Conjecture, states that if G is an n-vertex graph in which at least $n/2$ of the vertices have degree at least $n/2$, then G contains all trees with at most $n/2$ edges as subgraphs. Applying the Regularity Lemma, Ajtai, Komlós and Szemerédi [Graph theory, combinatorics, and algorithms, Vol. 2, 1135–1146, 1995] proved an approximate version of this conjecture. We prove it exactly for sufficiently large n. This immediately gives a tight upper bound for the Ramsey number of trees, and partially answers a conjecture of Burr and Erdős.

1. Introduction

For a graph G, let $V(G)$ (or simply V) and $E(G)$ denote its vertex set and edge set, respectively. The *order* of G is $v(G) = |V(G)|$, and the *size* of G is $e(G) = |E(G)|$ or simply $||G||$. For $v \in V$ and a set $X \subseteq V$, $N(v, X)$[1] represents the set of the neighbors of v in X, and $\deg(v, X) = |N(v, X)|$ is the degree of v in X. In particular $N(v) = N(v, V)$ and $\deg(v) = \deg(v, V)$.

Let G be a graph and T be a tree with $v(T) \le v(G)$. Under what condition must G contain T as a subgraph? Applying the greedy algorithm, one can easily derive the following fact.

Fact 1.1. *Every graph G with $\delta(G) = \min \deg(v) \ge k$ contains all trees T on k edges as subgraphs.*

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[1] We prefer $N(v, X)$ to the widely used notation $N_X(v)$ because we want to save the subscript for the underlying graph.

Extending Fact 1.1, Erdős and Sós [7] conjectured that the same holds when $\delta(G) \geq k$ is weakened to $a(G) > k-1$, where $a(G)$ is the average degree of G.

Conjecture 1.2 (Erdős-Sós). *Every graph on n vertices and more than $(k-1)n/2$ edges contains, as subgraphs, all trees with k edges.*

This celebrated conjecture was open till the early 90's, when Ajtai, Komlós and Szemerédi [1] proved an approximate version by using the celebrated Regularity Lemma of Szemerédi [16].

Another way to strengthen Fact 1.1 is replacing $\delta(G)$ by the median degree of G. The $k = n/2$ case of this direction was conjectured by Loebl [8] and became known as the $(n/2 - n/2 - n/2)$ Conjecture (see [9] page 44).

Conjecture 1.3 (Loebl). *If G is a graph on n vertices, and at least $n/2$ vertices have degree at least $n/2$, then G contains, as subgraphs, all trees with at most $n/2$ edges.*

The general case was conjectured by Komlós and Sós [8].

Conjecture 1.4 (Komlós - Sós). *If G is a graph on n vertices, and at least $n/2$ vertices have degree at least k, then G contains, as subgraphs, all trees with at most k edges.*

Conjecture 1.4 is trivial for stars and was verified by Bazgan, Li and Woźniak [3] for paths. Applying the Regularity Lemma, Ajtai, Komlós and Szemerédi proved [2] an approximate version of Conjecture 1.3.

Theorem 1.5 (Ajtai-Komlós-Szemerédi). *For every $\rho > 0$ there is a threshold $n_0 = n_0(\rho)$ such that the following statement holds for all $n \geq n_0$: If G is a graph on n vertices, and at least $(1+\rho)n/2$ vertices have degree at least $(1+\rho)n/2$, then G contains, as subgraphs, all trees with at most $n/2$ edges.*

The main goal of this paper is to prove Conjecture 1.3 *exactly* for sufficiently large n. Below we add floor and ceiling functions around $n/2$ to make the case when n is odd more explicit.

Theorem 1.6 (Main Theorem). *There is a threshold n_0 such that Conjecture 1.3 holds for all $n \geq n_0$. In other words, if G is a graph of order $n \geq n_0$, and at least $\lceil n/2 \rceil$ vertices have degree at least $\lceil n/2 \rceil$, then G contains, as subgraphs, all trees with at most $\lfloor n/2 \rfloor$ edges.*

It was shown in [2] that Conjecture 1.4 is best possible when $k+1$ divides n. But the sharpness of Conjecture 1.3 appears not to have been studied before. Clearly the $n/2$ as the degree condition cannot be weakened because T could be a star with $n/2$ edges. Is the other $n/2$, the number of large degree vertices, best possible? The following construction shows that this is essentially the case, more exactly, this $n/2$ cannot be replaced by $n/2 - \sqrt{n} - 2$.

Construction 1.7. *Let T be a tree with $n/2+1$ vertices distributed in 3 levels: the root has $n/4$ children, each of which has exactly one leaf. Let G be a graph such that $V(G) = V_1 + V_2$,*

$|V_1| = |V_2| = n/2$ *and each* $V_i = A_i + B_i$ *with* $|A_i| = n/4 - \sqrt{n}/2 - 1$*. Each vertex* $v \in A_i$ *is adjacent to all other vertices in* V_i *and exactly one vertex in* B_j *for* $j \neq i$*. The* $n/4 - \sqrt{n}/2 - 1$ *edges between* A_i *and* B_j *make up* $\sqrt{n}/2$ *vertex-disjoint stars centered at* B_j *of size either* $\sqrt{n}/2 - 1$ *or* $\sqrt{n}/2 - 2$*.*

Clearly the $n/2 - \sqrt{n} - 2$ vertices in $A_1 \cup A_2$ have degree $n/2$. We claim that G does not contain T. In fact, by symmetry in G, we only consider two possible locations for the root r of T: A_1 or B_1. Suppose that r is mapped to some $u \in B_1$. Since $\deg(u) \leq |A_1| + \sqrt{n}/2 - 1 = n/4 - 2$, there is no room for the $n/4$ children of r. Suppose that r is mapped to some $u \in A_1$. Let m be the size of a largest family of paths of length 2 sharing only u (*u-2-paths*). There are two kinds of u-2-paths containing *no* vertices from $A_1 \setminus \{u\}$: u to B_1 to A_2, and u to B_2 to A_2. Since the size of a maximal matching between B_1 and A_2 is $\sqrt{n}/2$ and $\deg(u, B_2) = 1$, we conclude that $m \leq |A_1| - 1 + \sqrt{n}/2 + 1 = n/4 - 1$. Hence there is no room for the $n/4$ 2-paths in T.

Define $\ell(G) = |\{u \in V(G) : \deg(u) \geq v(G)/2\}|$. Denote by $\mathcal{T}_k$ the set of trees on k edges. We write $G \supset \mathcal{T}_k$ when the graph G contains *all* members of $\mathcal{T}_k$ as subgraphs. Conjecture 1.4 actually consider the following extremal problem. Let $m(n,k)$ be the smallest m such that every n-vertex graph G with $\ell(G) \geq m$ contains all trees on k edges, *i.e.*, $G \supset \mathcal{T}_k$. Conjecture 1.4 says that $m(n,k) \leq n/2$ for all $k < n$, in particular, Conjecture 1.3 says that $m(n, n/2) \leq n/2$. Theorem 1.6 confirms that $m(n, n/2) \leq n/2$ for $n \geq n_0$ while Construction 1.7 shows that $m(n, n/2) > n/2 - \sqrt{n} - 2$. At present, we do not know the exact value of $m(n, n/2)$ or $m(n,k)$ for most values of k.

While studying an extremal problem on graphs, researchers are also interested in the structure of graphs whose size is close to the extreme value. Let $\mathrm{ex}(n, F)$ be the usual Turán number of a graph F. The stability theorem of Erdős-Simonovits [15] from 1966 proved that n vertex graphs without a fixed subgraph F with close to $\mathrm{ex}(n, F)$ edges have similar structures: they all look like the extremal graph. In this paper, though we can not determine $m(n, n/2)$ exactly, we are able to describe the structure of n-vertex graphs G with $\ell(G)$ about $n/2$ and $G \not\supset \mathcal{T}_{n/2}$. For simplicity on notations, Theorem 1.9 below only deals with the case when $v(G)$ is even though a similar result holds when $v(G)$ is odd.

Definition 1.8. *The half-complete graph* H_n *is a graph on* n *vertices with* $V = V_1 + V_2$ *such that* $|V_1| = \lfloor n/2 \rfloor$ *and* $|V_2| = \lceil n/2 \rceil$*. The edges of* H_n *are all the pairs inside* V_1 *and between* V_1 *and* V_2*. In other words,* $H_n = K_n - E(K_{\lceil n/2 \rceil})$*.*

For a graph G and $k \in \mathbf{N}$, we denote by kG the graph that consists of k *disjoint* copies of G, in other words, $V(kG)$ has a partition $\cup_{i=1}^{k} V_i$ such that its induced subgraph on each V_i is isomorphic to G.

Theorem 1.9 (Stability Theorem)**.** *For every* $\beta > 0$ *there exist* $\zeta > 0$ *and* $n_0 \in \mathbf{N}$ *such that the following statement holds for all* $n \geq n_0$*: if a* $2n$*-vertex graph* G *with* $\ell(G) \geq (1 - \zeta)n$

does not contain some $T \in \mathcal{T}_n$, then $G = 2H_n \pm \beta n^2$, i.e., *G can be transformed to two vertex-disjoint copies of H_n by changing at most βn^2 edges.*

The structure of the paper is as follows. In the next section we discuss the application of Theorem 1.6 on graph Ramsey theory. In Section 4 we state the Regularity Lemma and give some properties of regular pairs. In Section 3 we outline the proof of Theorem 1.6, comparing it with the proof of Theorem 1.5, and define two extremal cases. The details are given in Sections 5 and 6. In Section 5 we extend the ideas in [2] to prove the non-extremal case, where Subsection 5.5 contains most of our new ideas and many technical details. The extremal cases are covered in Section 6, in which we also complete the proof of Theorem 1.9. The last section contains some concluding remarks.

Notations: Let $[n] = \{1, 2, \ldots, n\}$. We may write $A + B$ instead of $A \cup B$ when two sets A and B are disjoint. Let $G = (V, E)$ be a graph. If $U \subset V$ is a vertex subset, we write $G - U$ for $G[V \setminus U]$, the induced subgraph on $V \setminus U$. When $U = \{v\}$ is a singleton, we often write $G - v$ rather than $G - \{v\}$. For a subset $H \subset E$, we write $G - H$ for the subgraph of G obtained by removing all edges in H and all vertices v such that all incident edges of v in G are in H. Given two *not necessary disjoint* subsets A and B of V, $e(A, B)$ denotes the number of *ordered* pairs (a, b) such that $a \in A$, $b \in B$ and $\{a, b\} \in E$. The *density* $d(A, B)$ between A and B and the minimum degree $\delta(A, B)$ from A to B are defined as follows:

$$d(A, B) = \frac{e(A, B)}{|A||B|}, \quad \delta(A, B) = \min_{a \in A} \deg(a, B).$$

A forest F is a disjoint union of trees. We write $T \in F$ if the tree T is a component of F. Trees in this paper are always rooted (though the choice of the roots is arbitrary). Let $Rt(F)$ be the set of roots in F, and let $Rt(T)$ be the root of T. Then $v(F) = ||F|| + |Rt(F)|$. For any vertex x in a tree, the *parent* $p(x)$ is its neighbor on the unique path from the root to x and the set of *children* is $C(x) = N(x) \backslash p(x)$. We use $F(x)$ to represent the maximal subtree of F containing x but not $p(x)$. For a forest F, we partition its vertices by levels, namely, their distances to the roots: $Level_i(F)$ denotes the set of vertices whose distance to a root is i (*e.g.*, $Level_0(F) = Rt(F)$) and $Level_{\geq i}(F) = \bigcup_{j \geq i} Level_j(F)$. Write $F_{even} = \bigcup Level_i(F)$ for all even i, and $F_{odd} = \bigcup Level_i(F)$ for all odd i. For a tree T, $T_{even} \cup T_{odd}$ is the unique bipartition of $V(T)$. We define $Ratio(F) = |F_{odd}|/v(F)$.

For two graphs G and H, we write $H \to G$ if H can be embedded into G, *i.e.*, there is an injection $\phi : V(H) \to V(G)$ such that $\{\phi(u), \phi(v)\} \in E(G)$ whenever $\{u, v\} \in E(H)$. For $X \in V(H)$ and $A \subseteq V(G)$, $\phi(X)$ stands for the union of $\phi(x)$, $x \in X$. When $\phi : H \to G$ and $\phi(X) \subseteq A$, we write $X \to A$.

2. Ramsey number of trees

An immediate consequence of Theorem 1.6 is a tight upper bound for the *Ramsey number* of trees. The Ramsey number $R(H)$ of a graph H is the minimum integer n such that every 2-edge-coloring of K_n yields a monochromatic copy of H. Let T_n be a tree on n vertices. What can we say about upper bounds for $R(T_n)$?

It is easy to see that $R(T_n) \leq 4n-3$. In fact, every 2-edge-coloring of K_{4n-3} yields a monochromatic graph G on $4n-3$ vertices with at least $\frac{1}{2}\binom{4n-3}{2}$ edges. Since every graph with average degree d contains a subgraph whose minimal degree is at least $d/2$, G contains a subgraph G' with minimal degree at least $(4n-4)/4 = n-1$. By Fact 1.1, G' thus contains a copy of T_n.

Burr and Erdős made [5] the following conjecture[2] on $R(T_n)$.

Conjecture 2.1. *$R(T_n) \leq 2n-2$ for even n and $R(T_n) \leq 2n-3$ for odd n.*

Note that [9] page 18 says that Burr and Erdős conjectured that $R(T_n) \leq 2n-2$, and [13] says that Loebl conjectured $R(T_n) \leq 2n$.

The bounds in Conjecture 2.1 are best possible when letting T_n be a star S_n on n vertices. For example, when n is even, there exists an $(n-2)$-regular graph G_1 on $2n-3$ vertices, and consequently 2-edge-coloring K_{2n-3} with G_1 as the red graph contains no monochromatic copy of S_n.

It is easy to check that the Erdős-Sós Conjecture implies Conjecture 2.1. On the other hand, Conjecture 1.3 suggests a slightly weaker bound on $R(T_n)$. To see this, suppose a 2-edge-coloring partitions K_{2n-2} into two subgraphs G_1 and G_2. Then either G_1 contains at least $n-1$ vertices of degree at least $n-1$ or G_2 contains at least n vertices of degree at least $n-1$. Conjecture 1.3 thus implies that either G_1 or G_2 contains all trees of order n. Our main theorem (Theorem 1.6) therefore essentially confirm Conjecture 2.1 for large n.

Corollary 2.2. *If n is sufficiently large and T_n is a tree on n vertices, then $R(T_n) \leq 2n-2$.*

Given two graph H_1, H_2, the asymmetric Ramsey number $R(H_1, H_2)$ is the minimum integer n such that every 2-coloring (red, blue) of K_n yields a red H_1 or a blue H_2. Theorem 1.6 actually implies that $R(T', T'') \leq 2n-2$ for large n, where T', T'' are (not necessarily the same) trees on n vertices. Furthermore, if correct, the Komlós-Sós Conjecture implies that $R(T_n, T_m) \leq m+n-2$, where T_n, T_m are arbitrary trees on n, m vertices, respectively.

Finally, when the bipartition of T_n is known, Burr conjectured [4] a upper bound for $R(T_n)$ which implies Conjecture 2.1, in terms of the sizes of two partition sets t_1, t_2. See [4, 10, 11] for progress on this conjecture.

[2]This is a different conjecture from their well-known conjecture on Ramsey numbers for graphs with degree constraints.

3. Structure of our proofs

In this section we sketch the proofs of the main theorem and Theorem 1.9.

Let us first recall the proof of Theorem 1.5. Given T and G as in Theorem 1.5, the authors of [2] first prepared T and G: T is folded such that it looks like a *bi-polar* tree, namely, a tree having two vertices (called *poles*) under which all subtrees are small, and G is treated with the Regularity Lemma. Then they applied the Gallai–Edmonds decomposition to the reduced graph G_r and found two clusters A, B of large degree and a matching covering the neighbors of A and B in G_r. Finally they embedded the bi-polar version of T into $\{A, B\} \cup M$ and showed how to convert this embedding to an embedding of T.

The two ρ's in Theorem 1.5 are to compensate the following losses. Assume that ε, d, γ are some small positive numbers determined by ρ. After applying the Regularity Lemma with parameters ε, d, the degrees of the vertices of L are reduced by $(d+\varepsilon)n$. In addition, the regularity of a regular pair (A, B) only guarantees (by a corollary of Lemma 4.6) an embedding of a forest (consisting of small-size trees) of order $(1-\gamma)(|A|+|B|)$, instead of $|A|+|B|$. Clearly the above losses are unavoidable as long as the Regularity Lemma is applied. In other words, without these two ρ's, we can only expect to embed trees of size smaller than $v(G)/2$ by copying the proof of Theorem 1.5.

In order to prove Theorem 1.6 which contains no error terms, we have to study the structure of G more carefully and also consider the structure of T in order to find a series of sufficient conditions for embedding T in G. If none of these conditions holds, then G can be split into two equal parts such that between them, there exist either almost no edges or almost all possible edges. In such extremal cases, we show that all trees with n edges can be found in the original graph G without using the Regularity Lemma.

Without loss of generality, we may assume that the order of the host graph G is even. In fact, when $v(G) = 2n-1$, the assumption of Theorem 1.6 says that there are at least n vertices of degree at least n in G. After adding one isolated vertex to G, the new graph $\tilde{G}$ still has at least n vertices of degree at least n. If a tree (on n edges) can be found in $\tilde{G}$, then it must be a subgraph of G.

Given a graph G of order $2n$ and $0 \le \alpha \le 1$, we define two *extreme cases with parameter* α.

We say that G is in the Extremal Case 1 with parameter α if

(EC1): *$V(G)$ can be evenly partitioned into two subsets V_1 and V_2 such that $d(V_1, V_2) \ge 1-\alpha$.*

We say that G is in the Extremal Case 2 with parameter α if

(EC2): *$V(G)$ can be evenly partitioned into two subsets V_1 and V_2 with $d(V_1, V_2) \le \alpha$.*

Note that if G is in **EC1** (or **EC2**) with parameter α, then G is in **EC1** (or **EC2**) with parameter x for any $x \le \alpha$.

We will show that $G \supset \mathcal{T}_n$, *i.e.*, G containing all trees on n edges if $\ell(G) \geq n$ and G is in either of the extremal cases.

Proposition 3.1. *For any $0 < \sigma < 1$, there exist $n_0 \in \mathbf{N}$ and $0 < c < 1$ such that the following holds. If $n \geq n_0$ and G is a $2n$-vertex graph with $\ell(G) \geq 2\sigma n$ and $V(G) = V_1 + V_2$ with $|V_1| = |V_2|$ and $d(V_1, V_2) \geq 1 - c$, then $G \supset \mathcal{T}_n$.*

Theorem 3.2. *There exist $\alpha > 0$ and $n_0 \in \mathbf{N}$ such that if $n \geq n_0$ and G is a $2n$-vertex graph with $\ell(G) \geq n$ and $V(G) = V_1 + V_2$ such that $|V_1| = |V_2|$ and $d(V_1, V_2) \leq \alpha$, then $G \supset \mathcal{T}_n$.*

To prove Theorem 1.6, we only need the $\sigma = 1/2$ case of Proposition 3.1. But Theorem 1.9 need the $\sigma < 1/2$ case. The core step in our proof is the following theorem, which describes the structure of G with $\ell(G) \geq (1 - \varepsilon)n$ and $G \not\supset \mathcal{T}_n$.

Theorem 3.3. *For every $\alpha > 0$ there exist $\varepsilon > 0$ and $n_0 \in \mathbf{N}$ such that the following statement holds for all $n \geq n_0$: if a $2n$-vertex graph G with $\ell(G) \geq (1 - \varepsilon)n$ does not contain some $T \in \mathcal{T}_n$, then G is in either of the two extreme cases with parameter α.*

Similarly, to prove Theorem 1.6, we only need the $\varepsilon = 0$ case of Theorem 3.3. This more general statement of Theorem 3.3 is necessary for the proof of Theorem 1.9 and becomes useful if one wants to show that $G \supset \mathcal{T}_n$ under a (slightly) smaller value of $\ell(G)$.

Proof of Theorem 1.6. Let G be a graph of order $2n$ for sufficiently large n. Assume that $G \not\supset \mathcal{T}_n$. Applying Proposition 3.1 with $\sigma = 1/2$, we know that G is not in the Extremal Case 1 with parameter α_1 for some $0 < \alpha_1 < 1$. Applying Theorem 3.2, we know that G is not in the Extremal Case 2 with parameter α_2 for some $0 < \alpha_2 < 1$. Finally we apply Theorem 3.3 with $\alpha = \min\{\alpha_1, \alpha_2\}$ and conclude that G is in either of the two extremal cases with parameter α, contradiction. $\square$

Theorem 1.9 easily follows from Proposition 3.1, Theorem 3.3, and Lemma 6.3 from Section 6.2, where Lemma 6.3 is also the main step in proof of Theorem 3.2.

4. Regular pairs and the Regularity Lemma

In this section we recall the Regularity Lemma and state some properties of regular pairs.

Definition 4.1. *Let $\varepsilon > 0$. A pair (A, B) of disjoint vertex-sets in G is ε-*regular *if for every $X \subseteq A$ and $Y \subseteq B$, satisfying $|X| > \varepsilon|A|$, $|Y| > \varepsilon|B|$, we have $|d(X, Y) - d(A, B)| < \varepsilon$.*

We use the following version of the Regularity Lemma:

Lemma 4.2 (Regularity Lemma - Degree Form)**.** *For every $\varepsilon > 0$ there is an $M(\varepsilon)$ such that if $G = (V, E)$ is any graph and $d \in [0, 1]$ is any real number, then there is a partition of the vertex set V into $\ell + 1$ partition sets (called clusters) $V_0, V_1, \ldots, V_\ell$, and there is a subgraph G' of G with the following properties:*

- $\ell \leq M(\varepsilon)$,
- $|V_0| \leq \varepsilon|V|$; *all clusters* V_i, $i \geq 1$, *are of the same size* $N \leq \varepsilon|V|$,
- $\deg_{G'}(v) > \deg_G(v) - (d+\varepsilon)|V|$ *for all* $v \in V$,
- V_i, $i \geq 1$, *is an independent set in* G',
- *all pairs* (V_i, V_j), $1 \leq i < j \leq \ell$, *are* ε*-regular in* G', *each with density either 0 or greater than* d.

Like many other problems to which the Regularity Lemma is applied, it suffices to consider the subgraph $G'' = G' - V_0$ as the underlying graph except for the extremal cases. We therefore skip the subscript G'' unless we consider G'' and G at the same time. Let $V' = V \setminus V_0$ denote the vertex set of $V(G'')$.

Given two vertex sets X and Y, recall that the minimum degree $\delta(X,Y)$ is $\min_{v\in X} \deg(v,Y)$. We now define the average degree from X to Y as

$$\overline{\deg}(X,Y) = \frac{1}{|X|} e(X,Y) = d(X,Y)\,|Y|.$$

Note the *asymmetry* of $\delta(X,Y)$ and $\overline{\deg}(X,Y)$. When $X = \{v\}$, we have $\overline{\deg}(v,Y) = \deg(v,Y)$. Finally we let $\overline{\deg}(X) = \overline{\deg}(X,V')$.

Denote the family of clusters $V_1,\ldots,V_\ell$ by $\mathcal{V}$ and use capital letters X,Y,A,B for clusters (elements of $\mathcal{V}$). For $X,Y \in \mathcal{V}$, if $d(X,Y) \neq 0$, *i.e.*, $d(X,Y) > d$, then we write $X \sim Y$ and call $\{X,Y\}$ a *non-trivial* regular pair.

Definition 4.3. *After applying the Regularity Lemma to* G, *we define the reduced graph* G_r *as follows: the vertices are* $1 \leq i \leq \ell$, *which correspond to clusters* V_i, $1 \leq i \leq \ell$, *and for* $1 \leq i < j \leq \ell$ *there is an edge between* i *and* j *if* $V_i \sim V_j$.

For a cluster $X = V_i \in \mathcal{V}$, we may abuse our notation by writing $\deg_{G_r}(X)$ or $N(X)$ instead of $\deg_{G_r}(i)$ or $N_{G_r}(i)$. The degree of X, $\overline{\deg}(X)$ and $\deg_{G_r}(X)$ have the following relationship

$$\overline{\deg}(X) = \frac{1}{|X|} e(X,V) = \sum_{Y\in\mathcal{V}, Y\sim X} d(X,Y)N \leq \sum_{Y\in\mathcal{V}, Y\sim X} N = \deg_{G_r}(X)\,N. \tag{1}$$

Definition 4.4. *Given clusters* $A,B \in \mathcal{V}$, *and a family* $\mathcal{Y} = \{Y \subseteq X : X \in \mathcal{S}\}$, *where* $\mathcal{S} \subseteq \mathcal{V}$ *is a family of clusters. A vertex* $u \in A$ *is called* typical *(atypical otherwise) to a set* $Y \subseteq B$ *if* $\deg(u,Y) > (d(A,B)-\varepsilon)|Y|$. *A vertex* $u \in A$ *is called* typical *(atypical otherwise) to (the family)* $\mathcal{Y}$ *if* u *is typical to all but at most* $\sqrt{\varepsilon}|\mathcal{Y}|$ *members of* $\mathcal{Y}$.

One immediate consequence of (A, B) being regular is that all but at most $\varepsilon|A|$ vertices $u \in A$ are typical to any subset Y of B with $|Y| > \varepsilon|B|$. In the following proposition, Part 1 says that for any $A \in \mathcal{V}$ and family $\mathcal{Y} = \{Y \subseteq V_i : V_i \in \mathcal{V}, |Y| > \varepsilon N\}$, most vertices in A are typical to $\mathcal{Y}$. As a corollary of Part 1, Part 2 says that the degree of a cluster is about the same as the degree of most vertices in the cluster.

Proposition 4.5. *Suppose that $V_1, V_2, \ldots, V_\ell$ are obtained from Lemma 4.2 and $n' = |V'|$. Let $i_0 \in [\ell]$, $I \subseteq [\ell] \setminus \{i_0\}$ and $Y_I = \cup_{i\in I} Y_i$, where each Y_i is a subset of V_i containing at least εN vertices. For every $u \in V_{i_0}$ we define*

$$I_u = \{i \in I : \deg(u, Y_i) \le (d(V_{i_0}, V_i) - \varepsilon)|Y_i|\}.$$

Then the following statements hold:

1. All but at most $\sqrt{\varepsilon}N$ vertices $u \in V_{i_0}$ satisfy $|I_u| \le \sqrt{\varepsilon}|I|$.

2. All but at most $\sqrt{\varepsilon}N$ vertices $u \in V_{i_0}$ satisfy $\deg(u, Y_I) > \overline{\deg}(V_{i_0}, Y_I) - 2\sqrt{\varepsilon}n'$. All but at most $\sqrt{\varepsilon}N$ vertices $u \in V_{i_0}$ satisfy $\deg(u, Y_I) < \overline{\deg}(V_{i_0}, Y_I) + 2\sqrt{\varepsilon}n'$.

Proof. *Part 1.* Suppose instead, that $|\{u \in V_{i_0} : |I_u| > \sqrt{\varepsilon}|I|\}| > \sqrt{\varepsilon}N$. Then

$$\sum_{i\in I} |\{u \in V_{i_0} : i \in I_u\}| = \sum_{u\in V_{i_0}} |I_u| > \sqrt{\varepsilon}N\sqrt{\varepsilon}|I| = \varepsilon N|I|.$$

Therefore we can find $i_1 \in I$ such that $|S| > \varepsilon N$ for $S = \{u \in V_{i_0} : i_1 \in I_u\}$. By the definition of I_u, we have

$$d(S, Y_{i_1}) = \sum_{u\in S} \frac{\deg(u, Y_{i_1})}{|S||Y_{i_1}|} \le d(V_{i_0}, V_{i_1}) - \varepsilon,$$

which contradicts the regularity between V_{i_0} and V_{i_1}.

Part 2. For every $u \in V_{i_0}$,

$$\begin{aligned}
\deg(u, Y_I) &\ge \sum_{i\notin I_u} \deg(u, Y_i) > \sum_{i\notin I_u} (d(V_{i_0}, V_i) - \varepsilon)|Y_i| > \sum_{i\notin I_u} (d(V_{i_0}, Y_i) - 2\varepsilon)|Y_i| \\
&= \sum_{i\in I} d(V_{i_0}, Y_i)|Y_i| - \sum_{i\in I_u} d(V_{i_0}, Y_i)|Y_i| - 2\varepsilon \sum_{i\notin I_u} |Y_i| \\
&\ge \overline{\deg}(V_{i_0}, Y_I) - \sum_{i\in I_u} |V_i| - 2\varepsilon|Y_I|.
\end{aligned}$$

According to Part I, all but $\sqrt{\varepsilon}N$ vertices of V_{i_0} further satisfy

$$\deg(u, Y_I) > \overline{\deg}(V_{i_0}, Y_I) - \sqrt{\varepsilon}N|I| - 2\varepsilon|Y_I| > \overline{\deg}(V_{i_0}, Y_I) - 2\sqrt{\varepsilon}n'.$$

The second claim can be proved similarly. □

One advantage of a regular pair is that regardless of its density, it behaves like a complete bipartite graph when we embed many small trees in it. This follows from repeatedly applying the following fundamental lemma, which gives an *online* embedding algorithm (embedding vertices one by one). Let us first introduce a notation to represent the flexibility of such an embedding. Suppose that an algorithm embeds the vertices of a graph H_1 one by one into another graph H_2. For a vertex $x \in V(H_1)$, a real number $p \neq 0$ and a set $A \subseteq V(H_2)$, we write $x \stackrel{p}{\to} A$ to indicate the flexibility of the embedding. When $p > 0$, it means that (at the moment when we consider x), our algorithm allows at least p vertices of A to be the image of x. When $p = -q < 0$, it means that all but at most q vertices of A can be chosen as the image of x. Such a flexibility is needed in Lemma 5.7 when we connect several forests into a tree.

Lemma 4.6. *Let $\varepsilon < d < \gamma$ be positive numbers such that $\gamma \geq \frac{4\varepsilon}{d-\varepsilon}$. Let N be an integer such that $\varepsilon N \geq 1$. Suppose that $\{X, Y\}$ is an ε-regular pair with $|X| = |Y| = N$ and $d(X, Y) > d$. Let $X_0, X_1 \subset X$, $Y_1 \subset Y$ satisfying $|X_0| \geq 3\varepsilon N$, $|X_1| \geq \gamma N$, $|Y_1| \geq \gamma N$. Then for any tree T of order εN with root r, there exists an online algorithm embedding $V(T)$ into $X_1 \cup Y_1$ such that $r \stackrel{2\varepsilon N}{\to} X_0$, $x \stackrel{2\varepsilon N}{\to} X_1$ for every $x \in T_{even} \setminus \{r\}$, and $y \stackrel{2\varepsilon N}{\to} Y_1$ for every $y \in T_{odd}$.*

Proof. Let $D_i = Level_i(T)$. We will sequentially embed $D_0, D_1, \ldots, D_s$ into $X_1 \cup Y_1$. First we embed r to a typical vertex $u \in X_0$ such that $\deg(u, Y_1) \geq (d(X,Y) - \varepsilon)|Y_1|$. Since at most εN vertices of A are atypical to Y_1 and $|X_0| \geq 3\varepsilon N$, at least $2\varepsilon N$ vertices of X_0 can be chosen as u.

Suppose that $D_0, \ldots, D_j$ are embedded (for example, say, D_j is embedded into X_1) such that $\deg(\phi(x), Y_1') > (d - \varepsilon)|Y_1'|$ for every $x \in D_j$, where ϕ denotes the embedding function and X_1', Y_1' denotes the set of unoccupied vertices. Now we consider the vertices in D_{j+1} in any order. Suppose that $y \in D_{j+1}$ has its parent $x \in D_j$. We embed y into an available vertex $u \in N(\phi(x), Y_1')$ which is typical to X_1', *i.e.*, $\deg(u, X_1') > (d - \varepsilon)|X_1'|$. If this is possible, this process may continue for all levels. By regularity between X and Y, at most εN vertices in Y_1 are atypical to X_1' (here we need $|X_1'| \geq \gamma N - \varepsilon N > \varepsilon N$). On the other hand, at most $|D_{j+1}| - 1$ vertices of Y_1' may already be occupied. The following inequality thus guarantees that at least $2\varepsilon N$ vertices can be chosen as u:

$$(d - \varepsilon)|Y_1'| - (|D_{j+1}| - 1) - \varepsilon N \geq 2\varepsilon N.$$

It suffices to have $(d - \varepsilon)|Y_1| \geq v(T) + 3\varepsilon N$. This holds because $|Y_1| \geq \gamma N$, $|V(T)| \leq \varepsilon N$ and $\gamma \geq \frac{4\varepsilon}{d-\varepsilon}$. □

5. The non-extremal case

The purpose of this section is to prove Theorem 3.3. We use the following parameters:

$$0 < \varepsilon \ll \gamma \ll d \ll \eta \ll \rho \ll \alpha \ll 1, \tag{2}$$

where $a \ll b$ can be specified as, for example, $100a < b^8$. Throughout our proof, *we assume that n is sufficiently large and omit floors and ceilings.* Let $G = (V, E)$ be a $2n$-vertex graph with $\ell(G) \geq (1-\varepsilon)n$, *i.e.*, at least $(1-\varepsilon)n$ vertices of degree at least n. We assume that G is *not* in **EC1** or **EC2** with parameter α. We apply the Regularity Lemma (Lemma 4.2) to G, and obtain the subgraph G'' and the reduced graph G_r.

The rest of the proof is divided into five subsections. In Section 5.1 we prove G'' and G_r have similar properties to G. In Section 5.2 we partition a tree T into a forest F such that all trees in $F - Rt(F)$ are small. In Section 5.3 we present a series of technical lemmas which give sufficient conditions for embedding F and correspondingly T into G''. Some of these lemmas (or their variations) appeared in [2] with very brief proofs. The reason why we state and (re)prove them is to make them applicable under new assumptions (the readers who are familiar with [2] may skip this subsection first). In Section 5.4 we prove a Tutte-type one-factor lemma, which states that the condition $\ell(G) \geq (1-\varepsilon)n$ forces a large matching in G_r. Since **EC1** does not hold in G, this suffices for embedding a tree of size near n into G''. The last subsection, Section 5.5, carefully checks case by case when we can even embed a tree of size n and conclude that **EC2** is the only exception.

5.1. Preparation of G

The goal of this subsection is to prove Claim 5.1, which gives the properties of G'' and G_r. Before stating the Lemma, we need the following preliminaries. Let L be the set of vertices in G of degree at least n. We call these *large* vertices, and call vertices in $V \setminus L$ *small* vertices. Since deleting edges between small vertices does not change our assumption, we assume that there is no edge between any two small vertices.

We may also assume that the number ℓ of clusters is even and equals to $2k$. In fact, if ℓ is odd, then eliminate one cluster by moving all the vertices in this cluster to V_0. In the worst case, we have $|V_0| \leq 2\varepsilon|V| = 4\varepsilon n$ and consequently $|V'| = 2Nk \geq 2n - 4\varepsilon n$. This implies that

$$n - 2\varepsilon n \leq Nk \leq n \tag{3}$$

We call a cluster **large** if it contains $2\sqrt{d}N$ large vertices (though the reason we set the threshold as $2\sqrt{d}N$ can only be seen in the proof of Claim 5.19). The set of large clusters is denoted by $\mathcal{L}$. We delete all the edges of G between two small clusters and thus assume every (non-trivial) regular-pair (of clusters) contains at least one large cluster.

Claim 5.1. *1. For every $X \in \mathcal{L}$, we have $\overline{\deg}(X) > n - 4dn$, and all but at most $\sqrt{\varepsilon}N$ vertices in X have degree in G'' greater than $n - 5dn$.*

2. $|\bar{\mathcal{L}}| \geq |\mathcal{L}| \geq (1 - 4\sqrt{d})k$, where $\bar{\mathcal{L}} = \{i \in V(G_r) : \deg_{G_r}(i) \geq (1-4d)k\}$.

3. If **EC1** *does not hold in G, then G_r contains two large adjacent clusters.*

Proof. *Part 1.* Applying Proposition 4.5 Part 2 to X and $Y_I = V' \setminus X$, we know that all but at most $\sqrt{\varepsilon}N$ vertices $u \in X$ satisfy

$$\deg(u, V' \setminus X) < \overline{\deg}(X, V' \setminus X) + 2\sqrt{\varepsilon}|V'|.$$

Note that the underlying graph is G''. Since $\deg_{G''}(u) = \deg(u, V' \setminus X)$ and $\overline{\deg}(X) = \overline{\deg}(X, V' \setminus X)$, it follows that

$$\deg_{G''}(u) < \overline{\deg}(X) + 4\sqrt{\varepsilon}n. \tag{4}$$

Since $|X \cap L| \geq 2\sqrt{d}N > \sqrt{\varepsilon}N$, we let u be a vertex of $X \cap L$. The definitions of G'' and L imply that

$$\deg_{G''}(u) \geq \deg_G(u) - (d+\varepsilon)2n - |V_0| \geq n - (d+3\varepsilon)2n > n - 3dn, \tag{5}$$

where the last inequality holds because $\varepsilon \ll d$ from (2). By putting (4) and (5)together, we conclude that $\overline{\deg}(X) > (1-3d)n - 4\sqrt{\varepsilon}n > n - 4dn$. Proposition 4.5 Part 2 further implies that all but at most $\sqrt{\varepsilon}N$ vertices in X have degree in G'' at least $n - 4dn - 4\sqrt{\varepsilon}n > n - 5dn$.

Part 2. We first observe that $\mathcal{L} \subseteq \bar{\mathcal{L}}$: for any $X \in \mathcal{L}$, Part 1 says that $\overline{\deg}(X) \geq (1-4d)n$, which implies that $\deg_{G_r}(X) \geq (1-4d)n/N \geq (1-4d)k$ because of (1) and (3). Next we show that $|\mathcal{L}| \geq (1-4\sqrt{d})k$. In fact, from $|L| \geq (1-\varepsilon)n$ and the definition of $\mathcal{L}$, we have

$$n - 5\varepsilon n \leq |L| - |V_0| = |L \cap V'| \leq |\mathcal{L}|N + 2\sqrt{d}N\left(2k - |\mathcal{L}|\right),$$

or $(N - 2\sqrt{d}N)|\mathcal{L}| \geq n - 5\varepsilon n - 4\sqrt{d}Nk$, which implies that $|\mathcal{L}| \geq (1-4\sqrt{d})k$ because of (2) and (3). We therefore conclude that $|\bar{\mathcal{L}}| \geq |\mathcal{L}| \geq (1-4\sqrt{d})k$.

Part 3. Suppose instead, that $\mathcal{L}$ is an independent set in G_r. Let U_1 be the set of the vertices of G contained in all the large clusters, and $U_2 := V \setminus U_1$. For all $v \in U_1$, we have $\deg_{G''}(v, U_1) = 0$, which implies that $\deg_{G''}(v, U_2) = \deg_{G''}(v)$. By Part 1, at least $(1-\sqrt{\varepsilon})N$ vertices v in a large cluster satisfy $\deg_{G''}(v) > n - 5dn$. By using $|\mathcal{L}| \geq (1-4\sqrt{d})k$ from Part 2, we have

$$e_{G''}(U_1, U_2) > (n-5dn)(1-\sqrt{\varepsilon})N|\mathcal{L}| \geq (n-5dn)(1-\sqrt{\varepsilon})N(1-4\sqrt{d})k > (1-10\sqrt{d})n^2.$$

Since $|U_1| = |\mathcal{L}|N \geq (1-4\sqrt{d})kN > (1-5\sqrt{d})n$, we can move at most $5\sqrt{d}n$ vertices from U_2 to U_1 such that $|U_1| = n$. The resulting sets U_1, U_2 satisfy

$$e_G(U_1, U_2) \geq e_{G''}(U_1, U_2) > (1-10\sqrt{d})n^2 - 5\sqrt{d}n^2 > (1-\alpha)n^2$$

since $d \ll \alpha$. This contradicts our assumption that **EC1** does not hold. □

5.2. Partition a tree into a forest

In this subsection we describe a way to cut a tree into a forest F in which all subtrees below $Rt(F)$ are small. We need the following definition. Recall that $T(x)$ denotes the maximal subtree in a rooted tree T containing a vertex x but not its parent $p(x)$.

Definition 5.2. *Let m be an integer.*

- *A tree T with root r is called an m-tree if $v(T(x)) \leq m$ for every $x \neq r$.*
- *A forest F is called an m-forest if F is the union of m-trees. An ordered m-forest is an m-forest with an ordered $Rt(F)$, in other words, it is a sequence of m-trees.*
- *Given a vertex x in a (rooted) tree T, we call $T(x)$ an m-subtree if $v(T(x)) > m$ and $v(T(y)) \leq m$ for every $y \in C(x)$.*

Note that every component of an m-forest is an m-tree – a tree whose size may be much larger than m but any subtree not containing the root has at most m vertices.

Given a tree T, we first transform it to an ordered m-forest. We start with $F = \emptyset$ and apply the following bottom-up transformation to T. We first move all m-subtrees to F, one at a time (note that these subtrees are disjoint in T). We repeat removing m-subtrees in the remaining tree till at most m vertices remains in T (it is easy to see that any tree with more than m vertices must contain an m-subtree). Add the subtree of the remaining vertices to F. Label the trees in F by $T_1, \ldots, T_t$ in the reversing order that they were added to F (so the tree added at last is T_1). The roots of F form an ordered set $R_0 = \{v_1, \ldots, v_t\}$ with $v_i = Rt(T_i)$, in particular, $v_1 = Rt(T)$. Since except for T_1, every tree in F has at least $m+1$ vertices, it follows that $t \leq \frac{v(T)-1}{m+1} + 1 = \frac{v(T)+m}{m+1}$. Let $u_i = p(v_i)$ be the parents of v_i in T.

We now refine F as follows. We call a vertex in F even (or odd) if the distance from it to $Rt(T)$ in T is even (or odd), for example, v_1 is even. We call two roots $v_i, v_j \in R_0$, $i < j$, *linked* if the parent u_j of v_j is a vertex of T_i. Later when we embed F into G, we want two linked roots to have the same parity unless one is the parent of the other one. To achieve this, we now cut the subtree $T_i(u_j)$ from T_i whenever two linked roots v_i, v_j have different parity and $u_j \neq v_i$. The new tree is inserted right before T_j in F; the new root u_j has the same parity as v_i.

Let $R = \{r_1, \ldots, r_s\}$ be the set of roots in the resulting F, with subsets R_a and R_b of the even roots and the odd roots, respectively. We have $|R_a|, |R_b| \leq |R_0|$ because, for example, each vertex of R_a is either an even vertex from R_0 or the parent of some odd vertex in R_0. Let $p_i = p(r_i)$ for $i \geq 2$ and call them *parent-vertices* (note that p_i are not necessarily distinct). In summary, $F = \{T_1, T_2, \ldots, T_s\}$ satisfies the following conditions:

(1) All T_i are m-trees, and $|R_a|, |R_b| \leq (v(T) + m)/(m+1)$.

(2) For $j = 2, \ldots, s$, we have $p_j \in T_i$ for some $i < j$.

(3) If $r_i \in R_a$ and $p_j \in T_i$, then either $p_j = r_i$ or $r_j \in R_a$. The same holds when we replace R_a by R_b.

It is easy to restore T from F. We start with T_1 and add $T_2, \ldots, T_s$ sequentially such that r_i is a child of p_i for $i \geq 2$. The order in F needs to be preserved when restoring T (in fact, after trees containing parent-vertices are connected together, the remaining trees can be added in any order).

5.3. Sufficient conditions for embedding trees

In this subsection we present a series of technical lemmas which give sufficient conditions for embedding trees into G'' (and thus in G).

Let C, X, Y be three distinct clusters in $\mathcal{V}$ with $X \sim Y$. Let F be an ordered εN-forest, which means that $Rt(F)$ is ordered, and all trees in the forest $F - Rt(F)$ are of order at most εN. We write $F \to (C, \{X, Y\})$ if there exists an online algorithm embedding the trees of F following the order of $Rt(F)$ such that $Rt(F) \to C$ and $F - Rt(F) \overset{2\varepsilon N}{\to} \{X, Y\}$, which means that $v \overset{2\varepsilon N}{\to} X$ or $v \overset{2\varepsilon N}{\to} Y$ for every $v \in V(F) \setminus Rt(F)$.

Given an εN-forest F, our first lemma gives sufficient conditions for $F \to (C, \{X, Y\})$. The most general case, Part 1, was proved in [2] and sufficed for their purpose. Recall that $||F||$ denotes the number of edges in a forest F , and the ratio of a tree T is $|T_{odd}|/v(T)$.

Lemma 5.3. *Let C, X, Y be three distinct clusters in $\mathcal{V}$ with $X \sim Y$. Write $d_x = d(C, X)$, $d_y = d(C, Y)$, and assume that $0 \le d_x \le d_y$. Let F be an ordered εN-forest with s roots for some $s \le \varepsilon N$.*

1. *If $||F|| \le (d_x+d_y-2\gamma-2\varepsilon)N$, then $F \to (C, \{X, Y\})$ such that $Rt(F)$ can be embedded to any s vertices in C that are typical to both X and Y.*
2. *Suppose that every tree in $F - Rt(F)$ has ratio between c and $1 - c$ (inclusively) for some $0 \le c \le \frac{1}{2}$. Then $F \to (C, \{X, Y\})$ in which $Rt(F)$ can be embedded to any s vertices in C that are typical to both X and Y if*

$$||F|| \le (2d_x - 2\gamma - 3\varepsilon)N + \frac{1}{1-c}(d_y - d_x)N \tag{6}$$
$$= (d_x + d_y - 2\gamma - 3\varepsilon)N + \frac{c}{1-c}(d_y - d_x)N.$$

3. *Suppose that $\lambda \le \{d_x, d_y\} \le 1 - \lambda$ for some $0 \le \lambda \le \frac{1}{2}$, and every tree in $F - Rt(F)$ contains at least two vertices. If $||F|| \le (d_x+d_y+\lambda-2\gamma-13\varepsilon)N$, then $F \to (C, \{X, Y\})$ such that each $r \in Rt(F)$ can be embedded a vertex in C that is typical to X^* and Y^*, the subsets of available vertices in X and Y when considering r.*

Proof. We present proofs of Part 1 and Part 2 here, and leave the proof of Part 3 to the appendix due to its complexity.

For Parts 1 and 2, we map the roots of F to typical vertices $u \in C$ such that $\deg(u, X) > (d_x - \varepsilon)N$ and $\deg(u, Y) > (d_y - \varepsilon)N$. This is possible because at most $2\varepsilon N$ vertices of C are atypical to either X or Y and $|C| = N > 2\varepsilon N + s$.

Let $F^o = F - Rt(F)$. Following the order of $Rt(F)$, we may regard F^o as a sequence $\{T_1, \dots, T_t\}$ such that the $T_1, \dots, T_{i_1}$ are under the first root, $T_{i_1+1}, \dots, T_{i_2}$ are under the second root of F, etc. Since F is an εn-forest, each T_i has at most εN vertices. We claim it suffices to show that $V(F^o)$ has a bipartition (A, B) satisfying the following properties. For simplicity, let $A_i = A \cap (V(T_1) \cup \dots \cup V(T_i))$ and $B_i = B \cap (V(T_1) \cup \dots \cup V(T_i))$. There exists $i_0 \le t$ such that

(I). $|A|, |B| \le (d_y - \gamma)N$,

(II). $|A_i|, |B_i| \le (d_x - \gamma)N$ for $i \le i_0$,

(III). $Rt(T_i) \in B$ for $i > i_0$.

Suppose that such a bipartition (A, B) exists, our goal is to embed trees $T_1, \dots, T_t$ sequentially such that $A \to X$ and $B \to Y$ as follows. Suppose that $T_1, \dots, T_{i-1}$ have been embedded and let us consider T_i. Suppose that $z = Rt(T_i)$ is adjacent to a vertex $r \in Rt(F)$ and r has been embedded to a typical vertex $u \in C$. Let X^*, Y^* denote the set of available vertices in X, Y, respectively, and P the set of available vertices in $N(u, X)$ (in $N(u, Y)$) if $z \in A$ ($z \in B$). In order to embed T_i by Lemma 4.6, we need to verify that $|X^*|, |Y^*| \ge \gamma N$ and $|P| \ge 3\varepsilon N$. From (I), $|A|, |B| \le (d_y - \gamma)N \le (1 - \gamma)N$, we immediately obtain that $|X^*|, |Y^*| \ge \gamma N$. When $i \le i_0$, since u is typical to X and Y, by (II), we have

$$|P| \ge \begin{cases} \deg(u, X) - |A_i| > (d_x - \varepsilon)N - (d_x - \gamma)N > 3\varepsilon N & \text{if } P \subseteq X; \\ \deg(u, Y) - |B_i| > (d_y - \varepsilon)N - (d_x - \gamma)N > 3\varepsilon N & \text{if } P \subseteq Y. \end{cases}$$

When $i > i_0$, by (III), we have $|P| \ge \deg(u, Y) - |B| > (d_y - \varepsilon)N - (d_y - \gamma)N > 3\varepsilon N$. Finally, the embedding provided by Lemma 4.6 guarantees that $v \overset{2\varepsilon N}{\to} X$ or $v \overset{2\varepsilon N}{\to} Y$ for every $v \in V(T_i)$.

We now show that such a bipartition always exists under the hypothesis of Parts 1 and 2.

Part 1. We sequentially add the two partition sets of $T_1, \dots, T_t$ to two empty sets in a balanced way such that $||A_i| - |B_i|| < \varepsilon N$ and $|A_i| \ge |B_i|$, where (A_i, B_i) is the resulting bipartition of the first i trees. This is possible because every T_i has at most $\varepsilon N - 1$ vertices in each of its partition sets. Let i_0 be the largest index such that $|A_i| \le (d_x - \gamma)N$. We next add the remaining trees to (A_{i_0}, B_{i_0}) by putting their roots in B_{i_0}. Denote the resulting sets by A and B.

Clearly (II) and (III) hold for such a partition (A, B). We only need to verify (I): $|A|, |B| \le (d_y - \gamma)N$. We first observe that

$$|A_{i_0}| > (d_x - \gamma - \varepsilon)N, \quad \text{and} \quad |B_{i_0}| > (d_x - \gamma - 2\varepsilon)N. \tag{7}$$

In fact, if $|A_{i_0}| \le (d_x - \gamma - \varepsilon)N$, then $|A_{i_0+1}| \le (d_x - \gamma)N$ after adding the next tree T_{i_0+1} to (A_{i_0}, B_{i_0}), contradicting the maximality of i_0. Since $|A_i| - |B_i| < \varepsilon N$, it follows that $|B_{i_0}| \ge (d_x - \gamma - 2\varepsilon)N$.

If $i_0 = t$, then $|B| \le |A| < (d_x - \gamma)N \le (d_y - \gamma)N$. Otherwise $i_0 < t$. Since $|A_{i_0}| \ge (d_x - \gamma - \varepsilon)N$ and $|A| + |B| = v(F^o) = ||F|| \le (d_x + d_y - 2\gamma - 2\varepsilon)N$, we have $|B| \le (d_y - \gamma - \varepsilon)N$. On the other hand, $|B_{i_0}| \ge (d_x - \gamma - 2\varepsilon)N$ implies that $|A| \le (d_y - \gamma)N$.

Part 2. We follow the same bipartition as in Part 1. Again it suffices to show that $|A|, |B| \le (d_y - \gamma)N$. First consider the $i_0 = t$ case. Since $|A| + |B| = v(F^o) = ||F||$ and $0 \le |A| - |B| < \varepsilon N$, we have $|A| \le (||F|| + \varepsilon N)/2$. The assumption (6) implies that $||F|| \le (2d_x - 2\gamma - 3\varepsilon)N + 2(d_y - d_x)N = (2d_y - 2\gamma - 3\varepsilon)N$, which gives that $|B| \le |A| \le (d_y - \gamma - \varepsilon)N$.

When $i_0 < t$, (7) holds. Let $A' = A - A_{i_0}$ and $B' = B - B_{i_0}$. By (6) and (7), we have $|A'|+|B'| \le \frac{1}{1-c}(d_y - d_x)N$. Since (A', B') is a bipartition of a forest of trees of ratio between c and $1-c$, we have $\max\{|A'|, |B'|\} \le (1-c)(|A'| + |B'|)$. Consequently $\max\{|A'|, |B'|\} \le (d_y - d_x)N$. Together with $|B_{i_0}| \le |A_{i_0}| \le (d_x - \gamma)N$, we have $\max\{|A|, |B|\} \le (d_x - \gamma + d_y - d_x)N = (d_y - \gamma)N$. □

Definition 5.4. *1. A* cluster-matching *is a family $\mathcal{M}$ of disjoint regular pairs in $\mathcal{V}$. The set of the clusters covered by $\mathcal{M}$ is denoted by $V(\mathcal{M})$ (hence the* size *$|\mathcal{M}|$ of $\mathcal{M}$ is the half of $|V(\mathcal{M})|$).*

2. *For a cluster $A \in \mathcal{V}$, we define $\overline{\deg}(A, \mathcal{M}) = \sum_{X \in V(\mathcal{M})} \overline{\deg}(A, X)$ to be the (average) degree of A to $\mathcal{M}$.*

3. *For $e = \{X, Y\} \in \mathcal{M}$, a cluster A and a vertex u, we simply write $\overline{\deg}(A, e)$ as $\overline{\deg}(A, X) + \overline{\deg}(A, Y)$, $d(A, e)$ as $d(A, X) + d(A, Y)$, and $\deg(u, e)$ as $\deg(u, X) + \deg(u, Y)$.*

Let $\mathcal{M}$ be a cluster-matching, A be a cluster not in $V(\mathcal{M})$, F be an ordered εN-forest. We write $F \to (A, \mathcal{M})$ if there is an online algorithm embedding the trees in F to $A \cup \bigcup_{C \in V(\mathcal{M})} C$ in order such that $Rt(F) \overset{-2\sqrt{\varepsilon}N}{\to} A$ and $F - Rt(F) \overset{2\varepsilon N}{\to} \mathcal{M}$, which means that for each tree T in $F - Rt(F)$ there exists a pair $\{X, Y\} \in \mathcal{M}$ such that for each vertex $v \in V(T)$, either $v \overset{2\varepsilon N}{\to} X$, or $v \overset{2\varepsilon N}{\to} Y$.

Lemma 5.5. *Suppose that $\mathcal{M}$ is a cluster-matching of size m and A is a cluster not in $V(\mathcal{M})$. Let F be an ordered εN-forest with $|Rt(F)| \le \varepsilon N$. Then $F \to (A, \mathcal{M})$ if any of the following holds:*

1. *$||F|| \le \overline{\deg}(A, \mathcal{M}) - 3\gamma n$.*

2. *There exist constants $c, \lambda \ge 0$ such that $|d(A, X) - d(A, Y)| \ge \lambda$ for all $(X, Y) \in \mathcal{M}$, all trees in F have ratio between c and $1 - c$ (inclusively), and $||F|| \le \overline{\deg}(A, \mathcal{M}) + \frac{c}{1-c}\lambda N m - 3\gamma n$.*

3. *There exists $0 \le \lambda \le \frac{1}{2}$ such that $\lambda \le d(A, X) \le 1 - \lambda$ for all $X \in V(\mathcal{M})$, every tree in F has at least two vertices, and $||F|| \le \overline{\deg}(A, \mathcal{M}) + \lambda N m - 3\gamma n$.*

Proof. Following the corresponding part of Lemma 5.3, we define the capacity of an edge $e = \{X, Y\} \in \mathcal{M}$ (together with A) hosting εN-forests

$$w(e) := \begin{cases} \overline{\deg}(A, e) - 2(\gamma + \varepsilon)N & \text{for Part 1} \\ \overline{\deg}(A, e) - (2\gamma + 3\varepsilon)N + \frac{c}{1-c}\lambda N & \text{for Part 2} \\ \overline{\deg}(A, e) + (\lambda - 2\gamma - 13\varepsilon N)N & \text{for Part 3.} \end{cases} \tag{8}$$

In each part, because $\varepsilon < \sqrt{\varepsilon} \ll \gamma$, it suffices to show that $F \to (A, \mathcal{M})$ if

$$|F| \le \sum_{e \in \mathcal{M}} w(e) - (4\sqrt{\varepsilon} + \varepsilon)Nm. \tag{9}$$

Proofs for all three parts 3 are essentially the same. The only difference is that in Parts 1 and 2, the roots of F are embedded into the vertices of C that are typical to X and Y for some $(X, Y) \in \mathcal{M}$ while in Part 3 ,the roots are embedded into the vertices of C that are typical to X^* and Y^*, the sets of available vertices in X and Y. Below we illustrate how to prove Part 1.

Given $\mathcal{M}$, A as stated, suppose that F satisfies (9) with $w(e) = \overline{\deg}(A, e) - 2(\gamma + \varepsilon)N$. We prove $F \to (A, \mathcal{M})$ together with the following claim by induction on the number of trees in F. Denote by $F(e)$ the part of F embedded in $e \in \mathcal{M}$. Except for one edge which hosts vertices from the last tree of F, all non-empty $F(e), e \in \mathcal{M}$ satisfy

$$w(e) - \varepsilon N < v(F(e)) \le w(e). \tag{10}$$

First assume that F contains only one εN-tree T_1. We map its root to a vertex $u \in A$ that is typical to the cluster-set $V(\mathcal{M})$, that is, typical to at least $(1 - \sqrt{\varepsilon})|V(\mathcal{M})|$ clusters in $V(\mathcal{M})$. By Proposition 4.5, all but at most $\sqrt{\varepsilon}N$ vertices in A can be chosen as u. Let $\mathcal{M}^* \subseteq \mathcal{M}$ be the set of all $e \in \mathcal{M}$ such that u is typical to both ends of e. Then $|\mathcal{M} \setminus \mathcal{M}^*| \le \sqrt{\varepsilon}|V(\mathcal{M})| \le 2\sqrt{\varepsilon}m$. Let $F_1 := T_1 - r_1$. Viewing F_1 as a set of trees of size at most εN, we partition F_1 into subset $\{F_1(e) : e \in \mathcal{M}^*\}$ of trees such that all but at most one nonempty $F_1(e)$ satisfies (10), while the exceptional $F_1(\tilde{e})$ satisfies $0 < v(F_1(\tilde{e})) \le w(e) - \varepsilon N$. Since each tree in F_1 has at most εN vertices, if such a partition does not exist, then

$$\begin{aligned} ||F|| = v(F_1) &> \sum_{e \in \mathcal{M}^*} (w(e) - \varepsilon N) \ge \sum_{e \in \mathcal{M}} (w(e) - \varepsilon N) \; - 2N|\mathcal{M} \setminus \mathcal{M}^*| \\ &\ge \sum_{e \in \mathcal{M}} w(e) - \; \varepsilon Nm \; - (2N)2\sqrt{\varepsilon}m, \end{aligned}$$

contradicting (9). If such a partition $\{F_1(e) : e \in \mathcal{M}^*\}$ exists, then $v(F_1(e)) \le w(e)$ and we can apply Lemma 5.3 Part 1 to embed $F_1(e) \overset{2\varepsilon N}{\to} e$ for every $e \in \mathcal{M}^*$.

Suppose that our (stronger) claim holds for every εN-forest F with $s - 1$ trees for some $2 \le s \le \varepsilon N$. Let $F = \{T_1, \dots, T_s\}$ be an ordered ε-forest satisfying (9). Let $\tilde{F} = \{T_1, \dots, T_{s-1}\}$. By the induction hypothesis, $\tilde{F} \to (A, \tilde{\mathcal{M}})$ for some $\tilde{\mathcal{M}} \subset \mathcal{M}$ such that (10) holds for $\tilde{F}$ and all but at most one edge $e \in \tilde{\mathcal{M}}$, and the exceptional edge $\tilde{e} \in \tilde{\mathcal{M}}$ hosts vertices from T_{s-1}. If $v(\tilde{F}(\tilde{e})) > w(\tilde{e}) - \varepsilon N$, then T_s can be embedded into $A \cup (\mathcal{M} \setminus \tilde{\mathcal{M}})$ as above because

$$||T_s|| = ||F|| - \sum_{i=1}^{s-1} ||T_i|| \leq \sum_{e\in\mathcal{M}} w(e) - (\varepsilon + 4\sqrt{\varepsilon})Nm - \sum_{e\in\tilde{\mathcal{M}}} (w(e) - \varepsilon N)$$
$$\leq \sum_{e\notin\tilde{\mathcal{M}}} w(e) - (\varepsilon + 4\sqrt{\varepsilon})Nm', \tag{11}$$

where $m' = |\mathcal{M} \setminus \tilde{\mathcal{M}}|$. All but at most $\sqrt{\varepsilon}N + s$ vertices of A can be the image of $Rt(T_s)$. Now assume that $v(\tilde{F}(\tilde{e})) \leq w(\tilde{e}) - \varepsilon N$. In this case we can not ignore $\tilde{e}$ and simply embed T_s into $A \cup (\mathcal{M} \setminus \tilde{\mathcal{M}})$ as above because it may create yet another edge $\hat{e} \in \mathcal{M} \setminus \tilde{\mathcal{M}}$ with $0 < v(F(\hat{e})) \leq w(e) - \varepsilon N$. If $||T_s|| + v(\tilde{F}(\tilde{e})) \leq w(\tilde{e})$, then Lemma 5.3, Part 1, implies that $\tilde{F}(\tilde{e}) \cup T_s$ can embedded to $(A, \tilde{e})$ while $Rt(T_s)$ is mapped to any vertex $u \in A$ that is typical to both ends of e. All but at most $2\varepsilon N + s$ vertices of A can be chosen as u.

Otherwise we partition T_s into two εN-trees T_s' and T_s'' sharing the same root r_s such that $w(\tilde{e}) - \varepsilon N < v(\tilde{F}(\tilde{e})) + ||T_s'|| \leq w(\tilde{e})$. This is possible because all trees below r_s are of size at most εN. We first map r_s to a vertex $u \in A$ typical to both ends of $\tilde{e}$ and the set $\mathcal{M} \setminus \tilde{\mathcal{M}}$. All but at most $2\varepsilon N + s + \sqrt{\varepsilon}N < 2\sqrt{\varepsilon}N$ vertices of A can be chosen as u. Then $\tilde{F}(\tilde{e})$ and T_s' can embedded to $A \cup \tilde{e}$ by Lemma 5.3. We next embed T_s'' into $\mathcal{M} \setminus \tilde{\mathcal{M}}$ because $||T_s''|| \leq \sum_{e\notin\tilde{\mathcal{M}}} w(e) - (\varepsilon + 4\sqrt{\varepsilon})Nm'$, as seen in (11). □

We need the next Lemma for Section 5.5.2. Its proof is similar to those of Lemma 5.3 and Lemma 5.5. The difference is that a forest F is embedded into three layers (A, $\mathcal{C}$ and $\mathcal{M}$) in Lemma 5.6 Part 2, instead of two layers as in Lemma 5.5.

Let F by an ordered εN-forest, $\mathcal{M}$ be a cluster-matching, $\mathcal{C}$ be a cluster-set disjoint from $V(\mathcal{M})$, and A be a cluster not contained in $V(\mathcal{M}) \cup \mathcal{C}$. We write $F \to (A, \mathcal{C}, M)$ if there is an online algorithm embedding $V(F)$ to $A \cup \bigcup_{X\in\mathcal{C}\cup V(\mathcal{M})} X$ such that $Rt(F) \overset{-2\sqrt{\varepsilon}N}{\to} A$, $Level_1(F) \overset{\gamma N}{\to} \mathcal{C}$, and $Level_{\geq 2}(F) \overset{2\varepsilon N}{\to} \mathcal{M}$.

Lemma 5.6. *1. Let C be a cluster containing a subset $P \subseteq C$. Let $\mathcal{M}$ be a cluster-matching not containing C such that $d(C, e) > 0$ for all $e \in \mathcal{M}$. Suppose that F is a forest of trees of order at most εN such that $|Rt(F)| \leq |P| - (\varepsilon + \gamma)N$ and $||F|| \leq (1-\gamma)|\mathcal{M}|N$. Then F can embedded into $P \cup \bigcup_{X\in V(\mathcal{M})} X$ such that $Rt(F) \overset{\gamma N}{\to} P$ and $F - Rt(F) \overset{2\varepsilon N}{\to} \mathcal{M}$.*

2. Let $\mathcal{M}$ be a cluster-matching, $\mathcal{C}$ be a cluster-set disjoint from $V(\mathcal{M})$, and A be a cluster not in $V(\mathcal{M}) \cup \mathcal{C}$. Let $m = \min_{C\in\mathcal{C}} |\{e \in \mathcal{M} : d(C, e) > 0\}|$. If F is an ordered εN-forest with $|Rt(F)| \leq \varepsilon N$, $|Level_1(F)| \leq \overline{\deg}(A, \mathcal{C}) - 2\gamma|\mathcal{C}|N$, and $|Level_{\geq 2}(F)| \leq (1 - \gamma)mN$, then $F \to (A, \mathcal{C}, M)$.

Proof. **Part 1.** Suppose that $F = \{T_1, T_2, \ldots, T_t\}$. Consider a pair $\{X, Y\} \in \mathcal{M}$ and suppose that $d(C, X) > 0$. We first embed $Rt(T_1)$ into a vertex $u_1 \in P$ such that

$\deg(u_1, X) > (d(C, X) - \varepsilon)|X|$. Then we apply Lemma 4.6 to embed trees in $T_1 - Rt(T_1)$ into $X \cup Y$ such that $Level_1(T_1) \to N(u_1, X)$. We repeat this process with the remaining trees in F. Let X^* and Y^* denote the sets of unoccupied vertices in X and Y, respectively, right before embedding T_i. If $|X^*|, |Y^*| \geq \gamma N$, then we embed $Rt(T_i)$ to an unoccupied vertex $u_i \in P$ such that $\deg(u_i, X^*) > (d(C, X) - \varepsilon)|X^*| > 3\varepsilon N$. Since $|P| - t - \varepsilon N \geq \gamma N$, at least γN vertices of P can be u_i. We then embed $T_i - Rt(T_i)$ by Lemma 4.6 with $X_1 = X^*$, $Y_1 = Y^*$, and $X_0 = N(u_i, X^*)$. If at least one of $|X^*| < \gamma N$, or $|Y^*| < \gamma N$, then at least $(1-\gamma)N$ vertices of F are already embedded in $\{X, Y\}$ and we consider other pairs in $\mathcal{M}$. Since $||F|| \leq (1-\gamma)|\mathcal{M}|N$, there must exist $\{X, Y\} \in \mathcal{M}$ such that $|X^*|, |Y^*| \geq \gamma N$.

Part 2. Suppose that $F = \{T_1, T_2, \ldots, T_s\}$ has roots $r_1, \ldots, r_s$. Let $F_i = T_i - \{r_i\}$. We first embed each r_i into a vertex $a_i \in A$ that are typical to $\mathcal{C}$, namely, there exists $\mathcal{C}_i \subset \mathcal{C}$ of size at least $(1-\sqrt{\varepsilon})|\mathcal{C}|$ such that $\deg(a_i, C) > (d(A, C) - \varepsilon)N$ for every $C \in \mathcal{C}_i$. By Proposition 4.5, all but $\sqrt{\varepsilon}N + s < 2\sqrt{\varepsilon}N$ vertices of A can be chosen as a_i for any $i \leq s$. Pick a cluster $C \in \mathcal{C}_1$ and let $P = N(a_1, C)$. Since $v(F_1) \leq ||F|| \leq (1-\gamma)mN$, if $|Rt(F_1)| \leq |P| - (\varepsilon+\gamma)N$ then we embed the forest F_1 into $P \cup \mathcal{M}$ by Part 1. Otherwise we embed a subforest F_1' with $|Rt(F_1')| = |P| - (\varepsilon + \gamma)N$. In this case, we pick another cluster in $\mathcal{C}_1$ and repeat this process. After embedding F_1, we embed the other F_i similarly. To see why this procedure works, let us assume that after embedding $F_1, \ldots, F_{j-1}$, we can not embed F_j for some $j \leq s$. This means that either there is not enough room for $Rt(F_j)$ in $\{N(a_j, C) : C \in \mathcal{C}_j\}$, or no enough room for $F_j - Rt(F_j)$ in $\mathcal{M}$. The former implies that the total number of the roots from $F_i, i \leq j$ is greater than

$$\sum_{C \in \mathcal{C}_j} (\deg(a_j, C) - \gamma N - \varepsilon N) > \sum_{C \in \mathcal{C}_j} (d(A, C) - \gamma - 2\varepsilon)N > \overline{\deg}(A, \mathcal{C}) - 2\gamma|\mathcal{C}|N.$$

The latter implies that the total number of vertices in $F_i - Rt(F_i), i \leq j$, is greater than $(1-\gamma)mN$ because for fixed $C \in C_j$, every $e \in \mathcal{M}$ with $d(C, e) > 0$ can host at least $(1-\gamma)N$ vertices. But none of these two cases are possible according to our assumption on $|Level_1(F)|$ and $|Level_{\geq 2}(F)|$. ☐

The following lemma gives sufficient condition for $T \subseteq G$ based on the embedding of F_a and F_b. We say two cluster-matchings are *disjoint* if the sets of the clusters that they cover are disjoint.

Lemma 5.7. *Let T be a tree of order n and $F = F_a \cup F_b$ be the ordered εN-forest generated from T as in Section 5.2. Let A, B be two adjacent clusters of size N in G with subsets $A_0 \subseteq A$ and $B_0 \subseteq B$ such that $|A_0|, |B_0| \geq \sqrt{d}N$. Then T can be embedded into G with $Rt(F) \to A_0 \cup B_0$ if any of the following holds.*

1. *There are two disjoint cluster-matchings $\mathcal{M}_a$ and $\mathcal{M}_b$ from $\mathcal{V} \setminus \{A, B\}$ such that $F_a \to (A, \mathcal{M}_a)$ and $F_b \to (B, \mathcal{M}_b)$.*
2. *There is an edge-partition of F_a into two forests F_0 and F_1 such that $V(F_0) \cap V(F_1) \subseteq Rt(F_a)$. There are three disjoint cluster-matchings $\mathcal{M}_0$, $\mathcal{M}_1$ and $\mathcal{M}_b$ from $\mathcal{V} \setminus \{A, B\}$ such that $F_0 \to (A, \mathcal{M}_0)$, $F_1 \to (A, \mathcal{M}_1)$, and $F_b \to (B, \mathcal{M}_b)$.*

3. *There is an edge-partition of F_a into two forest F_0 and F_1 such that $V(F_0) \cap V(F_1) \subseteq Rt(F_a)$. There are a cluster-set $\mathcal{C} \subset \mathcal{V} \setminus \{A, B\}$ and three disjoint cluster-matchings $\mathcal{M}_0$, $\mathcal{M}_1$ and $\mathcal{M}_b$ from $\mathcal{V} \setminus (\{A, B\} \cup \mathcal{C})$ such that $F_0 \to (A, \mathcal{C}, \mathcal{M}_0)$, $F_1 \to (A, \mathcal{M}_1)$, and $F_b \to (B, \mathcal{M}_b)$.*

Proof. Suppose that $F(T) = \{T_1, \dots, T_s\}$ with $r_i = Rt(T_i)$ and let $p_i = p(r_i)$ be the parent of r_i in T. Let ϕ be the given embedding function of F_a and F_b (into $\mathcal{M}_a$, $\mathcal{M}_b$ or $\mathcal{M}_0$). We embed the trees in F_a and F_b by ϕ following the order of F. We need to modify ϕ slightly such that all $\phi(p_i)$ and $\phi(r_i)$ are adjacent under three given conditions. Suppose that p_j is contained in T_i for some $i < j$.

Part 1. Without loss of generality, assume that $T_i \in F_a$. Below we consider the two cases $T_j \in F_a$ and $T_j \in F_b$.

Case 1a: $T_j \in F_a$. Then the distance between r_i and p_j is odd (at least 1). Assume that the subtree of T_i containing p_j is embedded into $\{X, Y\}$, and $\phi(p_j) \in X$. Then $X \sim A$ since the ancestor of p_j in $Level_1(T_i)$ is also embedded into X. By the definition of $F_a \to (A, \mathcal{M}_a)$, at least $2\varepsilon N$ vertices can be chosen as $\phi(p_j)$. Since at most εN vertices from X are atypical to A_0, we can choose $\phi(p_j)$ to be a vertex $w \in X$ that is typical to A_0. Later when we embed r_j, by the definition of $F_a \to (A, \mathcal{M}_a)$, all but at most $2\sqrt{\varepsilon}N$ vertices of A can be chosen as $\phi(r_j)$. Since at most εN vertices of A are atypical to B_0, we can always choose a typical (to B_0) vertex from $N(w, A_0)$ as $\phi(r_j)$ because $|N(w, A_0)| \geq (d - \varepsilon)\sqrt{d}N > 2\sqrt{\varepsilon}N + \varepsilon N$.

Case 1b: $T_j \in F_b$. Then $p_j = r_i$ because of the way we constructed F. We map r_i to a vertex $v \in A_0$ typical to B_0 and then map r_j to an unoccupied vertex from $N(v, B_0)$ that is typical to A_0. This is possible because $|N(v, B_0)| \geq (d - \varepsilon)\sqrt{d}N > 2\sqrt{\varepsilon}N + \varepsilon N$.

Part 2. There are a few possibilities for T_j: $T_j \in F_0$, $T_j \in F_1$, $T_j \in F_b$, and T_j is split into F_0 and F_1. Among these cases, only the case when T_j is split into F_0 and F_1 is not covered in Part 1. If $T_i \in F_a$, then we follow Case 1a to embed p_j to a vertex in $w \in X$ that is typical to A_0; if $T_i \in F_b$, then we follow Case 1b to embed $p_j = r_i$ to a vertex $w \in B_0$ typical to A_0. Since $|N(w, A_0)| \geq (d - \varepsilon)\sqrt{d}N > 4\sqrt{\varepsilon}N + \varepsilon N$ in both cases, we can choose $\phi(r_j)$ to be a typical (to B_0) vertex in A_0 that accommodates both $F_0 \to (A, \mathcal{M}_0)$ and $F_1 \to (A, \mathcal{M}_1)$.

Part 3. The new case is when $T_i \in F_0$. If $T_j \in F_b$, then $p_j = r_i$ and the situation is the same as Case 1b. Otherwise $T_j \in F_a$. Assume that the subtree of T_i containing p_j is embedded into $C \in \mathcal{C}$ and $\{X, Y\} \in \mathcal{M}_0$. Since the distance between r_i and p_j is odd, either $p_j \in Level_1(T_i)$ or $p_j \in Level_\ell(T_i)$ for some $\ell \geq 3$. Then either $\phi(p_j) \in C$ or (without loss of generality) $\phi(p_j) \in Y$. Then $X \sim C$ because the ancestor of p_j in $Level_2(T_i)$ is embedded into X. The case $\phi(p_j) \in C$ is the same as Case 1a. Now assume that $\phi(p_j) \in Y$. Let C^* be the set of unoccupied vertices in C. The definition of $x \overset{\gamma N}{\to} C$ for the last vertex x embedded in C implies that $|C^*| \geq \gamma N$. We first map the parent of p_j to be a vertex $v \in X$ that is typical to C^*. Next we map p_j to an unoccupied vertex $w \in N(v, C^*)$ that is typical to A_0 and X^*, the set of unoccupied vertices in X (this is needed to embed the children of p_j in T_i).

This is possible because $|N(v, C^*)| \geq (d-\varepsilon)\gamma N > 2\varepsilon N$. We finally map r_j to an unoccupied typical (to B_0) vertex from $N(w, A_0)$ because $|N(w, A_0)| \geq (d-\varepsilon)\sqrt{d}N > 2\sqrt{\varepsilon}N + \varepsilon N$. $\square$

The following simple fact was given in [2] without a proof.

Fact 5.8. *Let $\{a_i\}_{i=1}^m, \{b_i\}_{i=1}^m$ be two finite sequences such that $0 \leq a_i, b_i \leq \Delta$ for all i. Suppose that $\sum a_i = a$ and $\sum b_i = b$. Let s, t be positive real numbers such that $\frac{s}{a} + \frac{t}{b} \leq 1$. Then there is a partition of $[m]$ into I_1 and I_2 such that*

$$\sum_{i \in I_1} a_i > s - \Delta, \quad \textit{and} \quad \sum_{i \in I_2} b_i > t - \Delta.$$

Proof. We first reorder the two sequences such that $c_i = \frac{a_i}{a} - \frac{b_i}{b}$ is non-increasing. Then $\sum_{i=1}^j c_i \geq 0$ for any j because $\sum_{i=1}^m c_i = 0$. Choose $j \in [m]$ such that $s - \Delta < \sum_{i=1}^j a_i \leq s$. Then

$$\sum_{i>j} \frac{b_i}{b} = 1 - \sum_{i=1}^{j} \frac{b_i}{b} \geq 1 - \sum_{i=1}^{j} \frac{a_i}{a} \geq 1 - \frac{s}{a} \geq \frac{t}{b},$$

which gives $\sum_{i>j} b_i \geq t$. $\square$

Now we are ready to give sufficient conditions for embedding trees.

Lemma 5.9. *Let A and B be two adjacent clusters of size N with subsets $A_0 \subseteq A$ and $B_0 \subseteq B$ such that $|A_0|, |B_0| \geq \sqrt{d}N$. Let $\mathcal{M}$ be a cluster-matching on $\mathcal{V} \setminus \{A, B\}$. Given a tree T' of size at most n, let $F = F_a \cup F_b$ be the ordered εN-forest generated from T'. Then T' can be embedded with $Rt(F) \to A_0 \cup B_0$ if*

$$||F_a|| \leq \overline{\deg}(A, \mathcal{M}_a) - 3\gamma n \quad \textit{and} \quad ||F_b|| \leq \overline{\deg}(B, \mathcal{M}_b) - 3\gamma n, \tag{12}$$

in particular, when $||T'|| \leq \min\{\overline{\deg}(A, \mathcal{M}), \overline{\deg}(B, \mathcal{M})\} - 8\gamma n$.

Proof. If (12) holds, then we may apply Lemma 5.5 Part 1 to embed $F_a \to (A, \mathcal{M}_a)$ and $F_b \to (B, \mathcal{M}_b)$. By Lemma 5.7 Part 1, this implies that T' can be embedded to G with $Rt(F) \to A_0 \cup B_0$. Let $R_a = R_t(F_a)$ and $R_b = R_t(F_b)$. In order to apply Lemma 5.5, we only need to verify that $|R_a|, |R_b| \leq \varepsilon N$. Since $||T'|| \leq n$, by (2) and (3),

$$|R_a|, |R_b| \leq \frac{v(T) + \varepsilon N}{\varepsilon N + 1} \leq \frac{2n - 4\varepsilon n}{\varepsilon N} \leq \frac{2Nk}{\varepsilon N} \leq \frac{2M(\varepsilon)}{\varepsilon}, \tag{13}$$

where $M(\varepsilon)$ is the threshold in the Regularity Lemma. Since N is very large, $|R_a|$ and $|R_b|$ are much smaller than εN.

Now assume that $||T'|| \leq \min\{\overline{\deg}(A, \mathcal{M}), \overline{\deg}(B, \mathcal{M})\} - 8\gamma n$. Let $f_a = ||F_a||$ and $f_b = ||F_b||$. Then $f_a + f_b \leq ||T'||$. Let $s = f_a + 4\gamma n$ and $t = f_b + 4\gamma n$. Suppose that $\mathcal{M} = \{e_i\}$. Let

$a_i = \overline{\deg}(A, e_i)$, $b_i = \overline{\deg}(B, e_i)$, $a = \sum a_i$, and $b = \sum b_i$. We have $0 \le a_i, b_i \le \Delta := 2N$, and $a, b \ge ||T'|| + 8\gamma n$. Then

$$\frac{s}{a} + \frac{t}{b} \le \frac{f_a + 4\gamma n + f_b + 4\gamma n}{||T'|| + 8\gamma n} \le 1.$$

Fact 5.8 thus provides a partition of $\mathcal{M}$ into $\mathcal{M}_a$ and $\mathcal{M}_b$ such that $\overline{\deg}(A, \mathcal{M}_a) \ge f_a + 4\gamma n - 2N > f_a + 3\gamma n$, and similarly $\overline{\deg}(B, \mathcal{M}_b) \ge f_b + 3\gamma n$. The embedding of T' thus follows. □

5.4. Find a large matching in G_r

In this subsection we apply Tutte's one-factor theorem to prove Claim 5.11, which provides a matching in G_r that covers almost all the clusters we will consider. This lemma was proved in [2] without introducing the set O, whose role can only be seen in Stage 2 of Section 5.5, where we need the matching M to cover not only the neighbors of O but also the neighbors of $N(O) := \bigcup_{u \in O} N(u)$. When M is a matching and $u \notin V(M)$, we let $M^1(u) = \{(x, y) \in M : N(u, \{x, y\}) = 1\}$ and $M^2(u) = \{(x, y) \in M : N(u, \{x, y\}) = 2\}$.

Lemma 5.10. *Let H be a graph on $2k$ vertices and c be a real number such that $0 < c < 1$ and $ck \ge 1$. Suppose L is the set of vertices with degrees greater than $(1 - c)k$. If $|L| \ge (1 - c)k$ and L contains at least one edge, then there is either a matching in H covering all but $2ck + 1$ vertices or a matching M and two disjoint vertex subsets $S, O \subseteq V(H)$ such that*

- *$L \cap O$ contains two adjacent vertices,*
- *all but at most one vertex of $S \cup O$ are covered by M,*
- *for any $u \in O$, all but one vertex from $M^2(u)$ are also contained in O.*

Proof. We apply the Gallai–Edmonds decomposition to H. Let S denote the usual cut-set such that the following holds: every even component has a complete matching; every odd component O_i has a matching covering all but one vertex; and there is a matching $\{s_i x_i : i = 1, \dots, |S|\}$ such that $s_i \in S$ and $x_i \in O_i$. Let M be the union of these matchings. Below we show that unless $|V(M)| \ge 2k - 2ck - 1$, there is a component O outside S that contains two adjacent vertices of L. Then because $N(O) \subseteq O \cup S$, all but at most one vertex in $O \cup S$ are covered by M. In addition, for any $u \in O$ and any $xy \in M^2(u)$, we have $x, y \in O$, unless $x = x_i \in O$ and $y = s_i \in S$.

If no component contain any vertex of L, then $(1-c)k \le |L| \le |S|$. Consequently $|V(M)| \ge 2|S| \ge 2(1 - c)k$. On the other hand, if there are two components O_1, O_2 and two vertices $v_1, v_2 \in L$ such that $v_i \in O_i$, then $(1 - c)k \le \deg(v_i) \le |O_i| - 1 + |S|$ for $i = 1, 2$, then $|O_1| + |O_2| + 2|S| \ge 2(1 - c)k + 2$. Since each of O_1 and O_2 may contain one vertex not in M, we again have $|V(M)| \ge 2(1 - c)k + 2 - 2 = 2(1 - c)k$.

We may therefore assume there is a unique component O which contain all the vertices of $L\backslash S$. Let $a = |O \cap L|$, $b = |O \backslash L|$, we have $(1-c)k \le |L| \le a + |S|$. If there are two adjacent vertices in $O \cap L$, then we are done. Otherwise, for any vertex $v \in L \cap O$, we have $(1-c)k \le \deg(v) \le b + |S|$. Consequently $2|S| + |O| = 2|S| + a + b \ge (1-c)k + (1-c)k = 2(1-c)k$. Since at most one vertex in O may not be covered by M, we have $|V(M)| \ge 2(1-c)k - 1$. $\square$

We apply Lemma 5.10 to the reduced graph G_r and obtain the following claim.

Claim 5.11. *1. There are $A, B \in \mathcal{L} \cap \mathcal{O}$ with $A \sim B$.*

2. For any $U \in \mathcal{O}$, all but $9\sqrt{d}k$ neighbors of U are covered by $\mathcal{M}$.

3. For any $U \in \mathcal{O}$, all but one cluster from $\mathcal{M}^2(U)$ are also contained in $\mathcal{O}$.

Proof. Claim 5.1 Part 2 implies that the reduced graph G_r satisfies the condition of Lemma 5.10 with $L = \bar{\mathcal{L}}$ and $c = 4\sqrt{d}$. Applying Lemma 5.10, G_r either contains a matching covering all but at most $2(4\sqrt{d})k + 1 < 9\sqrt{d}k$ clusters, or a matching $\mathcal{M}$ satisfying the three properties of the lemma, which immediately proves our claim. In the former case, Claim 5.1 Part 3 guarantees that G_r contains two large adjacent clusters. By letting $\mathcal{O} = V(G_r)$, the three assertions follow. $\square$

Claim 5.1, Lemma 5.9, and Claim 5.11 together imply that any tree of size at most $(1 - 10\sqrt{d})n - 8\gamma n$ can be embedded into G. In fact, for any $X \in L \cap \mathcal{O}$, Claim 5.11 implies that $\overline{\deg}(X, \mathcal{M}) \ge \overline{\deg}(X) - 9\sqrt{d}kN$. By Claim 5.1, $\overline{\deg}(X) \ge (1-4d)n$ and consequently $\overline{\deg}(X, \mathcal{M}) \ge (1-4d)n - 9\sqrt{d}kN \ge (1 - 10\sqrt{d})n$. By Claim 5.11, there are two adjacent clusters $A, B \in \mathcal{L} \cap \mathcal{O}$. Then by Lemma 5.9, any tree of size at most $(1 - 10\sqrt{d})n - 8\gamma n$ can be embedded.

5.5. Embedding a tree of size n

In this subsection we finish the proof of Theorem 3.3.

Let T be a tree of size n. Recall that G is a $2n$-vertex graph satisfying $\ell(G) \ge (1-\varepsilon)n$ but neither **EC1** or **EC2** holds in G. Assume that T *cannot be embedded in G so our goal is to conclude a contradiction.*

Let $F = F_a \cup F_b$ be the εN-forest generated from T as in Section 5.2. Correspondingly $R = Rt(F)$ is partitioned into R_a and R_b satisfying (13). In particular, $c_f := |R|$ is a constant, much smaller than εN. Let $p_2, \ldots, p_{c_f}$ denote the parent-vertices and $f_a := ||F_a||$ and $f_b := ||F_b||$. Without loss of generality, assume that $f_a \ge f_b$. Since $f_a + f_b = ||F|| = n - c_f$, we have $f_b \le \frac{n}{2}$.

By Claim 5.11, there are two adjacent clusters A, B, a set $\mathcal{O}$ and a cluster-matching $\mathcal{M}$ such that $\overline{\deg}(A, \mathcal{M}), \overline{\deg}(B, \mathcal{M}) \ge (1 - 10\sqrt{d})n$. After removing some edges in G_r if necessary, we may assume that

$$\overline{\deg}(A, \mathcal{M}) = (1 - 10\sqrt{d})n, \quad \overline{\deg}(B, \mathcal{M}) = (1 - 10\sqrt{d})n. \tag{14}$$

Define $\mathcal{M}^+ = \{e \in \mathcal{M} : d(A,e) \geq d(B,e)\}$ and $\mathcal{M}^- = \mathcal{M} - \mathcal{M}^+$. Let $a^+ = \overline{\deg}(A, \mathcal{M}^+)$, $a^- = \overline{\deg}(A, \mathcal{M}^-)$, $b^+ = \overline{\deg}(B, \mathcal{M}^+)$ and $b^- = \overline{\deg}(B, \mathcal{M}^-)$. We thus have

$$a^+ \geq b^+, \quad b^- > a^-, \quad \text{and} \quad a^+ + a^- = b^+ + b^- = (1 - 10\sqrt{d})n.$$

Without loss of generality, we assume that $b^- \geq a^+$. Thus $b^- \geq b^+$ and consequently

$$b^- \geq (1 - 10\sqrt{d})n/2. \tag{15}$$

Our proof of Theorem 3.3 now splits into two stages. In Stage 1, we first show that unless f_b is very small, $d(A,e) \approx d(B,e)$ for most $e \in \mathcal{M}$. To facilitate the remaining part of the proof, we next give some sufficient conditions for embedding T into G, *e.g.*, there exist a sub-matching $\mathcal{M}_0 \subseteq \mathcal{M}$ such that significantly more than $\overline{\deg}(A, \mathcal{M}_0)$ vertices of F_a can be embedded into $\mathcal{M}_0$. By Lemma 5.5 Part 2, such an embedding exists if reasonably many pairs $\{X, Y\} \in \mathcal{M}$ have significantly different values for $d(A, X)$ and $d(A, Y)$, and reasonably many trees in $F_a - Rt(F_a)$ have ratio not close to 0 or 1. By Lemma 5.5 Part 3, such an embedding exists if reasonably many pairs $\{X, Y\} \in \mathcal{M}$ have the values of $d(A, X)$ and $d(A, Y)$ not close to 0 or 1, and reasonably many trees in $F_a - Rt(F_a)$ have at least two vertices. We then show that F has all desired structures. This implies that $d(A,X) \approx d(A,Y)$ and furthermore, $d(A,e) \approx 2$ or $d(A,e) \approx 0$ for most $e = \{X,Y\} \in \mathcal{M}$. Stage 1 is completed after we partition $\mathcal{V} = V(G_r)$ into $\mathcal{V}_1$ and $\mathcal{V}_2$ such that $\mathcal{V}_1$ is made up of the regular pairs $e \in \mathcal{M}$ in which $d(A,e) \approx 2$ (*e.g.*, $A, B \in \mathcal{V}_2$). Since the size of $\mathcal{V}_1$ is about the half of $|\mathcal{V}|$, the goal of Stage 2 is to show that $e_{G_r}(\mathcal{V}_1, \mathcal{V}_2)$ is small, which implies that **EC2** actually holds. In Stage 2, we first show that unless **EC2** holds, F_a basically consists of *root-2-paths*, paths of length 2 starting from some roots of F_a. Next we show such a special forest F_a (and F_b) can be easily embedded unless $e_{G_r}(\mathcal{S}_1, \mathcal{V}_2) \approx 0$, where $\mathcal{S}_1$ consists of the clusters in $\mathcal{V}_1$ whose partners are large clusters. Finally we show that $e_{G_r}(\mathcal{S}_1, \mathcal{V}_2) \approx 0$, which further implies $e_{G_r}(\mathcal{V}_1 \setminus \mathcal{S}_1, \mathcal{V}_2) \approx 0$. Consequently $e_{G_r}(\mathcal{V}_1, \mathcal{V}_2) \approx 0$, which implies that G is in **EC2**.

The main difference between Stage 1 and 2 is that in Stage 1, all the embeddings take place in A, B and regular pairs covering the neighbors of A and B, while in Stage 2, an embedding may take place on A, a set $\mathcal{C}$ of neighbors of A and regular pairs covering the neighbors of $\mathcal{C}$.

5.5.1. Stage 1

According to the Regularity Lemma, a (non-trivial) regular pair may have density any number between d and 1. Lemma 4.6 says that the actual density of a regular pair in $\mathcal{M}$ is irrelevant to our embedding, for example, we may assume that it equals to one. However, as seen in Lemma 5.5, the density of a regular pair (A, X) for a cluster $X \in \mathcal{M}$ determines the size of the embedding and may not assumed to be 1. In this subsection, we will show that under the assumption that $T \not\to G$, most of these regular pairs (A, X) have density close to one.

Recall that $a^+ - b^+ = b^- - a^- = \max_{\mathcal{M}'\subseteq\mathcal{M}} |d(A,\mathcal{M}') - d(B,\mathcal{M}')|$.

Claim 5.12. *If $f_b \geq d^{\frac{1}{4}}n$, then $a^+ - b^+ < 15d^{\frac{1}{4}}n$. Consequently $|d(A,\mathcal{M}') - d(B,\mathcal{M}')| < 15d^{\frac{1}{4}}n$ for* any *sub-matching $\mathcal{M}' \subseteq \mathcal{M}$.*

Proof. Suppose instead, that $f_b \geq d^{\frac{1}{4}}n$ and $a^+ - b^+ \geq 15d^{\frac{1}{4}}n$. We will show that $\mathcal{M}$ can be partitioned into $\mathcal{M}_a$ and $\mathcal{M}_b$ such that (12) holds, which implies $T \to G$, contradiction. Recall that $a^+ \leq b^-$. Then $b^- - b^+ = b^- - a^+ + a^+ - b^+ \geq 15d^{\frac{1}{4}}n$. Since $b^- + b^+ = (1-10\sqrt{d})n$ and $f_b \leq n/2$, we have $b^- \geq (1 + 15d^{\frac{1}{4}} - 10\sqrt{d})n/2 > f_b + 3\gamma n$.

We find a sub-matching $\mathcal{M}_b \subseteq \mathcal{M}^-$ by adding edges of $\mathcal{M}^-$ to an empty set in the descending order of $\frac{d(B,e)-d(A,e)}{d(B,e)}$ and stop as soon as $\sum d(B,e)N \geq f_b + 3\gamma n$. Then

$$f_b + 3\gamma n \leq \overline{\deg}(B,\mathcal{M}_b) \leq f_b + 3\gamma n + 2N. \tag{16}$$

Since $\frac{x_1}{y_1} \geq \frac{x_2}{y_2}$ implies that $\frac{x_1}{y_1} \geq \frac{x_1+x_2}{y_1+y_2}$ for any $x_1, x_2, y_1, y_2 > 0$, we have

$$\frac{\overline{\deg}(B,\mathcal{M}_b) - \overline{\deg}(A,\mathcal{M}_b)}{\overline{\deg}(B,\mathcal{M}_b)} \geq \frac{\overline{\deg}(B,\mathcal{M}^-) - \overline{\deg}(A,\mathcal{M}^-)}{\overline{\deg}(B,\mathcal{M}^-)} \geq \frac{15d^{\frac{1}{4}}n}{b^-}.$$

Consequently

$$\overline{\deg}(B,\mathcal{M}_b) - \overline{\deg}(A,\mathcal{M}_b) \geq \overline{\deg}(B,\mathcal{M}_b)\frac{15d^{\frac{1}{4}}n}{b^-} > f_b\,\frac{15d^{\frac{1}{4}}n}{n} \geq 15d^{\frac{1}{4}}d^{\frac{1}{4}}n = 15\sqrt{d}n,$$

where the last inequality uses the hypothesis $f_b > d^{\frac{1}{4}}n$.

Let $\mathcal{M}_a = \mathcal{M} - \mathcal{M}_b$. Then

$$\begin{aligned}
\overline{\deg}(A,\mathcal{M}_a) &= \overline{\deg}(A,\mathcal{M}) - \overline{\deg}(A,\mathcal{M}_b) \\
&\geq (1-10\sqrt{d})n + 15\sqrt{d}n - (f_b + 3\gamma n + 2N) \\
&= (n - f_b) + (15\sqrt{d}n - 10\sqrt{d}n - 3\gamma n - 2N) \\
&> f_a + 3\gamma n.
\end{aligned}$$

Thus $\mathcal{M}_a$ and $\mathcal{M}_b$ satisfy (12). By Lemma 5.9, we have $T \subset G$. □

We now give a few sufficient conditions for $T \subset G$. Roughly speaking, the following claim says that if there exists a reasonably large sub-matching $\mathcal{M}_0 \subseteq \mathcal{M}$, in which we can embed significantly more than $\deg(A,\mathcal{M}_0)$ vertices of F, then T can be embedded. Given a forest F, if two sub-forests F' and F'' satisfy that $E(F') \cup E(F'') = E(F)$ and $V(F') \cap V(F'') \subseteq Rt(F)$, then F' and F'' are called *root-sub-forests* of F.

Claim 5.13. *Given η as in (2), we have $T \subset G$ if any of the following conditions holds.*

1. *There exist a root-sub-forest* F_0 *of* F_a *and a sub-matching* $\mathcal{M}_0 \subset \mathcal{M}$ *such that* $F_0 \to (A, \mathcal{M}_0)$, $|\mathcal{M}_0| \le \eta k$, *and* $||F_0|| \ge \eta^3 n + \overline{\deg}(A, \mathcal{M}_0)$.

2. *There exist a root-sub-forest* F_0 *of* F_b *and a sub-matching* $\mathcal{M}_0 \subset \mathcal{M}$ *satisfies* $F_0 \to (B, \mathcal{M}_0)$, $|\mathcal{M}_0| \le \eta k$, *and* $|F_0| \ge \eta^3 n + \overline{\deg}(B, \mathcal{M}_0)$.

3. *Suppose that* $d \ll \varepsilon_1 \ll \varepsilon_2$. *There is a partition of* $\mathcal{M} = \mathcal{M}_{in} + \mathcal{M}_{out}$ *such that* $\overline{\deg}(A, \mathcal{M}_{in}) \ge (1 - \varepsilon_1)n$. *There are sub-matchings* $\mathcal{M}_0 \subset \mathcal{M}_{in}$, $\mathcal{M}_2 \subset \mathcal{M}_{out}$, *and a root-sub-forest* F_0 *of* F_a *such that*

$$|\mathcal{M}_0| + |\mathcal{M}_2| \le k/5, \quad F_0 \to (A, V(\mathcal{M}_0), \mathcal{M}_2), \quad \textit{and } ||F_0|| \ge \varepsilon_2 n + \overline{\deg}(A, \mathcal{M}_0).$$

Proof. Proofs for all three parts are similar. We give a full proof for Part 1 but for Parts 2 and 3, we only highlight differences between them and Part 1. Note that Part 2 is not a symmetric case of Part 1 because we lose the symmetry after we assumed that $f_a \ge f_b$.

Part 1. Let $F_1 = F_a - E(F_0)$. By Lemma 5.7 Part 2, it suffices to find a partition $\mathcal{M}_b \cup \mathcal{M}_1$ of $\mathcal{M} \setminus \mathcal{M}_0$ such that $F_b \to (B, \mathcal{M}_b)$ and $F_1 \to (A, \mathcal{M}_1)$.

Case 1a: $f_b \ge d^{\frac{1}{4}} n$. Since $|\mathcal{M}_0| \le \eta k$ and $f_b \le n/2$,

$$\overline{\deg}(B, \mathcal{M} \setminus \mathcal{M}_0) \ge (1 - 10\sqrt{d})n - 2N\eta k > f_b + 3\gamma n. \tag{17}$$

We thus find a sub-matching $\mathcal{M}_b$ of $\mathcal{M} \setminus \mathcal{M}_0$ such that (16) holds. By Lemma 5.5 Part 1, it implies that $F_b \to (B, \mathcal{M}_b)$. Let $\mathcal{M}_1 = \mathcal{M} \setminus (\mathcal{M}_b \cup \mathcal{M}_0)$. Since $f_b \ge d^{\frac{1}{4}} n$, Claim 5.12 implies that $|\overline{\deg}(A, \mathcal{M}_b) - \overline{\deg}(B, \mathcal{M}_b)| \le 15 d^{\frac{1}{4}} n$. By (16), this gives that $\overline{\deg}(A, \mathcal{M}_b) \le f_b + 3\gamma n + 2N + 15 d^{\frac{1}{4}} n$. Since $||F_0|| \ge \eta^3 n + \overline{\deg}(A, \mathcal{M}_0)$ and $||F_1|| + ||F_0|| + f_b \le n$, we obtain that

$$\begin{aligned}
\overline{\deg}(A, \mathcal{M}_1) &= \overline{\deg}(A, \mathcal{M}) - \overline{\deg}(A, \mathcal{M}_0) - \overline{\deg}(A, \mathcal{M}_b) \\
&\ge (1 - 10\sqrt{d})n - (||F_0|| - \eta^3 n) - (f_b + 3\gamma n + 2N + 15 d^{\frac{1}{4}} n) \\
&\ge ||F_1|| + 3\gamma n,
\end{aligned} \tag{18}$$

where in the last inequality we use $N \le \varepsilon n$ and (2). We thus have $F_1 \to (A, \mathcal{M}_1)$ by Lemma 5.5 Part 1.

Case 1b: $f_b < d^{\frac{1}{4}} n$. Note that $\mathcal{M}^-$ need not be disjoint from $\mathcal{M}_0$. By (15),

$$\overline{\deg}(B, \mathcal{M}^- \setminus \mathcal{M}_0) \ge (1 - 10\sqrt{d})n/2 - 2N\eta k > f_b + 3\gamma n, \tag{19}$$

we then choose a sub-matching $\mathcal{M}_b \subseteq (\mathcal{M}^- \setminus \mathcal{M}_0)$ such that (16) holds. Define $\mathcal{M}_1$ as in Case 1a and derive (18) similarly by using $\overline{\deg}(A, \mathcal{M}_b) \le \overline{\deg}(B, \mathcal{M}_b)$.

Part 2. Let $F_1 = F_b - E(F_0)$. We need to find a partition $\mathcal{M}_1 \cup \mathcal{M}_a$ of $\mathcal{M} \setminus \mathcal{M}_0$ such that $\overline{\deg}(B, \mathcal{M}_1) > ||F_1|| + 3\gamma n$ and $\overline{\deg}(A, \mathcal{M}_a) > ||F_a|| + 3\gamma n$. Since $f_b \ge ||F_0|| \ge \eta^3 n > d^{\frac{1}{4}} n$, we can follow the procedure in Case 1a of Part 1.

Part 3. We proceed as in Part 1 with $\mathcal{M} \setminus (\mathcal{M}_0 \cup \mathcal{M}_2)$ replacing $\mathcal{M} \setminus \mathcal{M}_0$ in Case 1a and $\mathcal{M}^- \setminus (\mathcal{M}_0 \cup \mathcal{M}_2)$ replacing $\mathcal{M}^- \setminus \mathcal{M}_0$ in Case 1b. Since $|\mathcal{M}_0| + |\mathcal{M}_2| \le k/5$, the second inequalities in (17) and (19) still hold. Let $\mathcal{M}_1 = \mathcal{M}_{in} \setminus (\mathcal{M}_b \cup \mathcal{M}_0)$. Since $\overline{\deg}(A, \mathcal{M}_{in}) \ge (1 - \varepsilon_1)n$ and $\gamma \ll d \ll \varepsilon_1 \ll \varepsilon_2$, the inequality (18) still holds. Lemma 5.7 Part 3 thus implies that $T \subset G$. □

From now on we may assume that none of the three conditions in Lemma 5.13 holds.

Let us consider the structure of F carefully. We first observe that there are not many leaves of F in $Level_1(F)$. Let us partition the set of leaves in $Level_1(F)$ into $W_1 \cup W_1'$, where W_1 is the set of the leaves of T located in $Level_1(F)$, and W_1' is the set of the parent-vertices that are leaves in $Level_1(F)$. We know that $|W_1'| \le c_f < \varepsilon N$. If $|W_1| \ge 11\sqrt{d}n$, then because of (14), $T - W_1$ can be embedded by Lemma 5.9 with A_0, B_0 as the set of large vertices in A, B, respectively (the definition of $\mathcal{L}$ implies that $|A_0|, |B_0| \ge 2\sqrt{d}N$). The vertices in W_1 can be added greedily at last. Thus assume that $|W_1| < 11\sqrt{d}n$. Let $\tilde{F} = F - W_1 - W_1'$. Then $\tilde{F}$ is a forest with no leaves in its first level. Then

$$||\tilde{F}|| \ge n - c_f - 11\sqrt{d}n - c_f > (1 - 12\sqrt{d})n \tag{20}$$

Define $\tilde{F}_a = F_a - W_1 - W_1'$. Since $||F_a|| \ge (n - c_f)/2$, we can derive that $||\tilde{F}_a|| > n/2 - 12\sqrt{d}n$.

Second we claim that reasonably many trees in $F - Rt(F)$ have ratio not close to 0 or 1. We first recall a simple fact on trees.

Fact 5.14. *Given a tree T, if $V(T)$ can be partitioned into a nonempty subset U_1 and an independent subset U_2, then U_2 contains at least $|U_2| - |U_1| + 1$ leaves. In particular, any tree with at least two vertices containing at least $||T_{even}| - |T_{odd}|| + 1$ leaves.*

Proof. Let a vertex $x \in U_1$ be the root (here we need $U_1 \ne \emptyset$). Let U_2' be the set of non-leaf vertices in U_2. Since each vertex in U_2' has at least one child in $U_1 \setminus \{x\}$ (using the fact that U_2 is independent) and the sets of children are disjoint, we have $|U_1| - 1 \ge |U_2'|$ and consequently the number of leaves in U_2 is at least $|U_2| - |U_1| + 1$. Now assume that $v(T) \ge 2$. Then both of its partition sets T_{even} and T_{odd} are nonempty. Letting U_2 be the larger set of T_{even} and T_{odd}, then U_2 contains at least $||T_{even}| - |T_{odd}|| + 1$ leaves. □

Recall that G is not in **EC1** with parameter $\alpha \gg d$.

Claim 5.15. *Let $\alpha_0 = \alpha/16$ and $F^2 = \{T \in F - Rt(F) : \alpha_0 < Ratio(T) < 1 - \alpha_0\}$. Then $v(F^2) > \alpha_0 n$.*

Proof. Let $F^1 := \tilde{F} \setminus F^2$. Then $v(F^1) + v(F^2) = ||\tilde{F}|| \ge (1 - 12\sqrt{d})n$ by (20). Suppose instead, that $v(F^2) \le \alpha_0 n$ and consequently $v(F^1) \ge (1 - 12\sqrt{d} - \alpha_0)n$.

Consider a tree $T \in F^1$. The definition of $\tilde{F}$ implies that $v(T) \ge 2$. By Fact 5.14, T contains at least $||T_{even}| - |T_{odd}|| + 1$ leaves. Since $Ratio(T) \le \alpha_0$ or $Ratio(T) \ge 1 - \alpha_0$, the tree T

has at least $(1-2\alpha_0)v(T)$ leaves. The total number of leaves in F^1 is thus at least

$$(1-2\alpha_0)(1-12\sqrt{d}-\alpha_0)n > (1-2\alpha_0)(1-2\alpha_0)n = (1-4\alpha_0)n + 4\alpha_0^2 n.$$

Since F is a decomposition of T and F contains at most c_f trees, F has at most $2c_f$ more leaves than T (the new leaves must be the roots of F or the parent-vertices). Since $c_f < \varepsilon N$, we have $4{\alpha_0}^2 n > 2c_f+1$. Then T has at least $(1-4\alpha_0)n+1$ leaves, or at most $4\alpha_0 n$ non-leaf vertices.

On the other hand, the set L of large vertices of G contains at least $(1-\varepsilon)n$ vertices. Let V_1 be a set of size n containing at least $(1-\varepsilon)n$ vertices of L. Let $L_1 := V_1 \cap L$. Since **EC1** does not hold in G, we have $d(V_1, V\setminus V_1) < 1-\alpha$. Consequently

$$e(L_1, V\setminus L_1) = e(L_1, V\setminus V_1) + e(L_1, V_1\setminus L_1) \le (1-\alpha)n^2 + \varepsilon n^2$$

and

$$e(L_1, L_1) > e(L, V) - e(L_1, V\setminus L_1) \ge (1-\varepsilon)n^2 - (1-\alpha+\varepsilon)n^2 > \alpha n^2/2.$$

This implies that the average degree of the induced subgraph $G[L_1]$ is at least $\alpha n/2$. By a well-known fact in graph theory, $G[L_1]$ has a subgraph G_0 such that $\delta(G_0) \ge \alpha n/4 = 4\alpha_0 n$. We may therefore embed all non-leaf vertices of T into G_0 using the greedy algorithm. Since the vertices in L_1 have degree at least n, we can add all the leaves to complete the embedding of T by the greedy algorithm. This contradicts our assumption that $T \not\to G$. □

Claim 5.15 and Lemma 5.5 Part 2 now lead to the following claim.

Claim 5.16. $|\mathcal{M}_1| < \eta k$, *where*

$$\mathcal{M}_1 = \{\{X, Y\} \in \mathcal{M} : |d(A, X) - d(A, Y)| \ge \eta\}.$$

Proof. Suppose instead, that there exists a sub-matching $\mathcal{M}_0 \subseteq \mathcal{M}_1$ of size ηk. Claim 5.15 showed that $v(F^2) \ge cn$. Let $F_a^2 = F^2 \cap F_a$ and $F_b^2 = F^2 \cap F_b$. First assume that $v(F_a^2) \ge \alpha_0 n/2$. Since $\frac{\alpha_0}{2}n > 2N\eta k + \eta^3 n \ge \overline{\deg}(A, \mathcal{M}_0) + \eta^3 n$, and any tree in F_a^2 has at most εN vertices, we can find a sub-forest $\hat{F}_0$ of F_a^2 such that

$$\overline{\deg}(A, \mathcal{M}_0) + \eta^3 n \le v(\hat{F}_0) < \overline{\deg}(A, \mathcal{M}_0) + \eta^3 n + \varepsilon N.$$

By adding the vertices in $Rt(F_a)$ adjacent to the roots of $\hat{F}_0$, we extend $\hat{F}_0$ to a root-sub-forest F_0 of F. Then $||F_0|| = v(\hat{F}_0)$. Since

$$||F_0|| < \overline{\deg}(A, \mathcal{M}_0) + \eta^3 n + \varepsilon N < \overline{\deg}(A, \mathcal{M}_0) + \alpha_0\eta N|\mathcal{M}_0| - 3\gamma n,$$

Lemma 5.5 Part 2 implies that $F_0 \to (A, \mathcal{M}_0)$, which gives rise to the first condition of Lemma 5.13. When $v(F_b^2) \ge cn/2$, the above arguments give rise to the second condition of Lemma 5.13. In either case we have a contradiction. □

By definition, every tree in $\tilde{F}_a - Rt(\tilde{F}_a)$ has at least two vertices. By using $||\tilde{F}_a|| \ge n/2 - 12\sqrt{d}n$, the following claim follows Lemma 5.5 Part 3.

Claim 5.17. $|\mathcal{M}_2| < \eta k$, *where*

$$\mathcal{M}_2 = \{\{X,Y\} \in \mathcal{M} : \eta \le d(A,X) \le 1-\eta \text{ and } \eta \le d(A,Y) \le 1-\eta\}.$$

Proof. Suppose instead, that there is a matching $\mathcal{M}_0 \subseteq \mathcal{M}_2$ of size ηk. Since $||\tilde{F}_a|| > 2N\eta k + \eta^3 n$ and $\tilde{F}_a$ is an εN-forest, we may find a root-sub-forest F_0 of $\tilde{F}_a$ (thus a root-sub-forest F_0 of F_a) such that

$$\overline{\deg}(A,\mathcal{M}_0) + \eta^3 n \le ||F_0|| < \overline{\deg}(A,\mathcal{M}_0) + \eta^3 n + \varepsilon N.$$

Since $\overline{\deg}(A,\mathcal{M}_0) + \eta^3 n + \varepsilon N < \overline{\deg}(A,\mathcal{M}_0) + (\eta N)|\mathcal{M}_0| - 3\gamma n$, Lemma 5.5 Part 2 implies that $F_0 \to (A,\mathcal{M}_0)$. This gives the first condition of Lemma 5.13, a contradiction. □

Let $\mathcal{M}_3 = \{\{X,Y\} \in \mathcal{M} \setminus \mathcal{M}_1 : d(A,X) < \eta \text{ or } d(A,Y) < \eta\}$. For each $\{X,Y\} \in \mathcal{M}_3$, we have $d(A,X) + d(A,Y) < 3\eta$ and consequently $\overline{\deg}(A,\mathcal{M}_3) < 3\eta Nk$.

Let $\mathcal{M}_{in} = \mathcal{M} - \mathcal{M}_1 - \mathcal{M}_2 - \mathcal{M}_3$. In other words,

$$\mathcal{M}_{in} = \{e \in \mathcal{M} : |d(A,X) - d(A,Y)| < \eta, \text{ either } d(A,X) > 1-\eta \text{ or } d(A,Y) > 1-\eta\}.$$

For every $e = \{X,Y\} \in \mathcal{M}_{in}$, we have

$$d(A,X), d(A,Y) > 1-2\eta \quad \text{and} \quad \overline{\deg}(A,e) > 2-3\eta. \tag{21}$$

Hence $\mathcal{M}_{in} \subseteq \mathcal{M}^2(A)$. Let $\mathcal{M}_{out} = \mathcal{M} - \mathcal{M}_{in}$. By Claim 5.11, at most one cluster in $V(\mathcal{M}_{in})$ may not be in $\mathcal{O}$. We move the edge containing this cluster to $\mathcal{M}_{out}$ if it exists. Together with Claim 5.16 and Claim 5.17, it gives

$$\overline{\deg}(A,\mathcal{M}_{in}) > (1-10\sqrt{d})n - \eta k2N - \eta k2N - 3\eta Nk - 2N > (1-8\eta)n. \tag{22}$$

By (21), we have $(2-3\eta)N|\mathcal{M}_{in}| < \overline{\deg}(A,\mathcal{M}_{in}) \le (1-10\sqrt{d})n$ and consequently

$$|\mathcal{M}_{in}| < (1+2\eta)k/2 \quad \text{and} \quad |\mathcal{V}_1| < (1+2\eta)k \tag{23}$$

5.5.2. Stage 2

Let V_i denote the set of vertices of G contained in the clusters in $\mathcal{V}_i$ for $i = 1,2$. By (22) and (23), we have $(1-8\eta)n \le |V_1| \le (1+2\eta)n$. If $e_{G_r}(\mathcal{V}_1,\mathcal{V}_2) \le \rho k^2$ for ρ satisfying (2), then

$$e_{G''}(V_1,V_2) \le dN^2|\mathcal{V}_1||\mathcal{V}_2| + \sum_{X\in\mathcal{V}_1, Y\in\mathcal{V}_2, X\sim Y} N^2 \le (\rho+d)n^2,$$

which implies that $e_G(V_1,V_2) < 2\rho n^2$. After adding or removing at most $8\eta n$ vertices to or from V_1 such that $|V_1| = |V_2| = n$, we still have $e(V_1,V_2) < 3\rho n^2$, which contradicts the assumption that **EC2** does not hold.

We therefore assume that $e_{G_r}(\mathcal{V}_1, \mathcal{V}_2) > \rho k^2$, which implies that reasonably many clusters in $\mathcal{V}_1$ have relatively large neighborhood in $\mathcal{V}_2$. If a cluster $X \in \mathcal{V}_1$ has many neighbors in $\mathcal{M}_{out}$, then we may use Lemma 5.6 Part 2 to embed a tree $T_i \in F_a$ into $A \cup X \cup \mathcal{M}_{out}$ such that $Rt(T_i) \to A$, $Level_1(T_i) \to X$ and $Level_{\geq 2}(T_i) \to \mathcal{M}_{out}$. When T_i has more than 3 vertices, this embedding is more efficient than embedding T_i into $A \cup \mathcal{M}_{in}$. If many trees in F_a have more than 3 vertices, then we obtain a sub-forest $\tilde{F}_a$ satisfying the third condition of Claim 5.13.

Claim 5.18. *Let* $F_3 = \{T \in F_a - Rt(F_a) : v(T) \geq 3\}$ *and* $\rho_0 = \rho/33$. *Then* $v(F_3) < 3\rho_0 n$.

Proof. Suppose instead, that $v(F_3) \geq 3\rho_0 n$. We will use Claim 5.13 Part 3 to embed a root-sub-forest F_0 for which $F_0 - Rt(F_0) \subseteq F_3$ into $(A, V(\mathcal{M}_0), \mathcal{M}_2)$ for some $\mathcal{M}_0 \subset \mathcal{M}_{in}$ and $\mathcal{M}_2 \subset \mathcal{M}_{out}$.

We partition $\mathcal{V}_2$ into three sets of size about $|\mathcal{V}_2|/3$ as follows: first evenly partition $\mathcal{M}_{out}$ into three sub-matchings $\mathcal{M}^1_{out}$, $\mathcal{M}^2_{out}$ and $\mathcal{M}^3_{out}$ and then add the clusters of $\mathcal{V}_2 \setminus V(\mathcal{M}_{out})$ into $V(\mathcal{M}^1_{out})$, $V(\mathcal{M}^2_{out})$ and $V(\mathcal{M}^3_{out})$ such that the sizes of three resulting sets differ by at most 2. By averaging, one of the partition sets, denoted by $\mathcal{V}_2'$, satisfies that

$$e(\mathcal{V}_1, \mathcal{V}_2') \geq e(\mathcal{V}_1, \mathcal{V}_2)/3 \geq \rho k^2/3 = 11\rho_0 k^2.$$

Let $\mathcal{V}_1'$ be the set of clusters $C \in \mathcal{V}_1$ such that $\deg_{G_r}(C, \mathcal{V}_2') \geq 9\rho_0 k$. Then $|\mathcal{V}_1'| \geq \rho_0 k$ (otherwise $e(\mathcal{V}_1, \mathcal{V}_2') < \rho_0 k|\mathcal{V}_2'| + |\mathcal{V}_1| 9\rho_0 k < 11\rho_0 k^2$). Let $\mathcal{C}$ be a subset of $\mathcal{V}_1'$ of size $\rho_0 k$. We have $\mathcal{C} \subset \mathcal{O}$ because $\mathcal{V}_1 \subseteq \mathcal{O}$. Let $C \in \mathcal{C}$. By Claim 5.11 Part 2, all but at most $9\sqrt{d}k$ neighbors of C are covered by $\mathcal{M}_{out}$. Let $\mathcal{M}_2$ be the sub-matching of $\mathcal{M}_{out}$ restricted on $\mathcal{V}_2'$. Then $\deg_{G_r}(C, V(\mathcal{M}_2)) > 9\rho_0 k - 9\sqrt{d}k \geq 8\rho_0 k$. Let $\mathcal{M}_0$ be the (minimum) sub-matching of $\mathcal{M}_{in}$ that covers $\mathcal{C}$. Then $|\mathcal{M}_0| \leq |\mathcal{C}| = \rho_0 k$, and $|\mathcal{M}_0| + |\mathcal{M}_2| < k/5$ because $|\mathcal{M}_2| \lesssim |\mathcal{V}_2|/6$.

Since $v(F_3) \geq 3\rho_0 n$, $\overline{\deg}(A, \mathcal{M}_0) \leq 2\rho_0 n$ and every tree in F_3 has at most εN vertices, we can find a root-sub-forest F_0 of F_a such that $F_0 - Rt(F_0) \subseteq F_3$ and

$$\overline{\deg}(A, \mathcal{M}_0) + \rho_0 n/2 \leq ||F_0|| < \overline{\deg}(A, \mathcal{M}_0) + \rho_0 n/2 + \varepsilon N.$$

It suffices to show that $F_0 \to (A, \mathcal{C}, \mathcal{M}_2)$ because then we can apply Claim 5.13 Part 3 with $\varepsilon_1 = 8\eta$ and $\varepsilon_2 = \rho_0/2$ to get $T \subset G$. Let $m = \min_{C \in \mathcal{C}} |\{e \in \mathcal{M}_2 : d(C, e) > 0\}|$. Then $m \geq \deg_{G_r}(C, V(\mathcal{M}_2))/2 > 4\rho_0 k$ and consequently $||F_0|| \leq (1-\gamma) mN$. Since every tree in $F_0 - Rt(F_0)$ has at least three vertices, $|Level_1(F_0)| \leq ||F_0||/3 \leq (5\rho_0 \frac{n}{2} + \varepsilon N)/3$. By (21) with $|\mathcal{C}| = \rho_0 k$, we have

$$\overline{\deg}(A, \mathcal{C}) - 2\sqrt{\varepsilon}|\mathcal{C}|N \geq (1 - 2\eta - 2\sqrt{\varepsilon}) N \rho_0 k \geq \frac{\frac{5}{2}\rho_0 n + \varepsilon N}{3} \geq |Level_1(F_0)|.$$

We may thus apply Lemma 5.6 Part 2 to obtain $F_0 \to (A, \mathcal{C}, \mathcal{M}_2)$. □

Recall that $\tilde{F}_a$ is the sub-forest of F_a obtained by removing all leaves in $Level_1(F_a)$, and $||\tilde{F}_a|| > n/2 - 12\sqrt{d}n$. A *root-2-path* in F is a path of length 2 having one end in $Rt(F)$.

Claim 5.18 thus leaves only one case unsettled, that is, when most vertices of $\tilde{F}_a$ are covered by root-2-paths. The rest of this section is devoted to this special case.

Let $\mathcal{S}_1 = \{Y : \{X, Y\} \in \mathcal{M}_{in}$ for some $X \in \mathcal{L}\}$, the set of clusters whose partners in $\mathcal{M}_{in}$ are large clusters. Since no regular pair runs between two small clusters, all the small clusters of $\mathcal{V}_1$ are contained in $\mathcal{S}_1$ (though $\mathcal{S}_1$ may contain large clusters as well). Let $\mathcal{L}_1 = \mathcal{V}_1 \setminus \mathcal{S}_1$. Since their partners in $\mathcal{M}_{in}$ are small clusters, all the clusters in $\mathcal{L}_1$ are large and located in different regular pairs of $\mathcal{M}_{in}$.

Claim 5.19. $e_{G_r}(\mathcal{S}_1, \mathcal{V}_2) < 16\rho k^2$.

Proof. We may assume that there are at least $10\rho k$ clusters in $\mathcal{S}_1$ each of which has at least $5\rho k$ neighbors in $\mathcal{V}_2$. Otherwise $e_{G_r}(\mathcal{S}_1, \mathcal{V}_2) < 10\rho k|\mathcal{V}_2| + |\mathcal{S}_1|5\rho k < 16\rho k^2$, and we are done. We pick $5\rho k$ such clusters that are located in different pairs of $\mathcal{M}_{in}$ and denote this cluster-set by $\mathcal{S}_0$. Let $\mathcal{M}_0$ be the minimum sub-matching of $\mathcal{M}_{in}$ covering $\mathcal{S}_0$. Let $\mathcal{L}_0 = V(\mathcal{M}_0) \setminus \mathcal{S}_0$ be the partner set of $\mathcal{S}_0$. We know that $\mathcal{L}_0 \subset \mathcal{L}$. Since $\deg(C, \mathcal{V}_2) \geq 5\rho k = |\mathcal{S}_0|$ for all $C \in \mathcal{S}_0$, we may choose different neighbors in $\mathcal{V}_2$ for each element of $\mathcal{S}_0$ to form a new matching $\mathcal{M}'_0$ that covers $\mathcal{S}_0$. Let $\mathcal{M}' = \mathcal{M}_{in} - \mathcal{M}_0 + \mathcal{M}'_0$.

By Claim 5.18, there are at least $(\frac{n}{2} - 12\sqrt{d} - 3\rho_0 n)/2 > n/8$ root-2-paths in F_a. We pick $4\rho n$ 2-paths which contain no parent-vertices (hence these paths may be embedded at any time). Let Z be the set of the mid-points and leaves of these paths. Then $|Z| = 8\rho n$. Let $T' = T - Z$ (then T' is a tree). Our plan is to embed T' into $A \cup B \cup V(\mathcal{M}')$ and then embed Z.

We claim that T' or its corresponding εN-forest F' satisfies the condition of Lemma 5.9. Here $F' = F'_a \cup F_b$ with $F'_a = F_a - Z$. First assume that $f_b \geq d^{\frac{1}{4}}n$. Then we have $|\overline{\deg}(A, \mathcal{M}_{in}) - \overline{\deg}(B, \mathcal{M}_{in})| \leq 15d^{\frac{1}{4}}n$ from Claim 5.12. Since $\overline{\deg}(A, \mathcal{M}_{in}) \geq (1 - 8\eta)n$, it follows that $\overline{\deg}(B, \mathcal{M}_{in}) \geq (1 - 8\eta - 15d^{\frac{1}{4}})n$. Then

$$\overline{\deg}(A, \mathcal{M}') \geq \overline{\deg}(A, \mathcal{M}_{in}) - \overline{\deg}(A, \mathcal{L}_0) \geq (1 - 8\eta - 5\rho)n, \tag{24}$$

and $\overline{\deg}(B, \mathcal{M}') \geq (1 - 8\eta - 15d^{\frac{1}{4}} - 5\rho)n$. Since $||T'|| = (1 - 8\rho)n$, we have $||T'|| \leq \min\{\overline{\deg}(A, \mathcal{M}'), \overline{\deg}(B, \mathcal{M}')\} + 8\gamma n$. Now assume that $f_b < d^{\frac{1}{4}}n$. Since $\overline{\deg}(B, \mathcal{M}^- \setminus \mathcal{M}_0) \geq (1 - 10\sqrt{d})n/2 - 2N5\rho k > f_b + 3\gamma n$, we can choose a sub-matching $\mathcal{M}_b \subseteq (\mathcal{M}^- \setminus \mathcal{M}_0)$ such that (16) holds. Define $\mathcal{M}_a = \mathcal{M}' - \mathcal{M}_b$. By (24), we have

$$\overline{\deg}(A, \mathcal{M}_a) = \overline{\deg}(A, \mathcal{M}') - \overline{\deg}(A, \mathcal{M}_0) \geq (1 - 5\rho - 8\eta)n - (f_b + 3\gamma n + 2N) > ||F'_a|| + 3\gamma n,$$

because $||F'_a|| = (1 - 8\rho)n - f_b$. Thus (12) holds with F'_a replacing F_a. We may therefore apply Lemma 5.9 to embed T'.

We next embed the mid-points in Z into the clusters of $\mathcal{L}_0$ and embed the leaves in Z at last by the greedy algorithm. Claim 5.1 says that in each large cluster, there are at least $2\sqrt{d}N$ *large* vertices, whose degrees are at least n, and at least $(1 - \varepsilon)N$ *typical* vertices, whose degrees are at least $(1 - 5d)n$. Note that a vertex can be both large and typical. For each

$X \in \mathcal{L}_0$, we construct two *disjoint* subsets $P_X, Q_X \subset X$ such that P_X consists of $2\sqrt{d}N$ large vertices and Q_X consists of $(1-2\sqrt{d}-\varepsilon)N$ typical vertices. By Proposition 4.5, at most $\sqrt{\varepsilon}N$ vertices of A are atypical to $\{P_X : X \in \mathcal{L}_0\}$; at most $\sqrt{\varepsilon}N$ vertices of A are atypical to $\{Q_X : X \in \mathcal{L}_0\}$. Let $A_0 \subset A$ consist of all large vertices that are typical to both $\{P_X : X \in \mathcal{L}_0\}$ and $\{Q_X : X \in \mathcal{L}_0\}$. Then $|A_0| \geq 2\sqrt{d}N - 2\sqrt{\varepsilon}N > \sqrt{d}N$. By Lemma 5.9, we can embed $Rt(F_a)$ to A_0 while embedding T'. This means if $u \in A_0$ is the image of a root in F_a, there exist subsets $\mathcal{L}_0', \mathcal{L}_0'' \subseteq \mathcal{L}_0$ such that $|\mathcal{L}_0'|, |\mathcal{L}_0''| \geq (1-\sqrt{\varepsilon})|\mathcal{L}_0|$ and

$$\deg(u, P_X) \geq (d(A,X)-\varepsilon)|P_X| \quad \text{for all } X \in \mathcal{L}_0',$$
$$\deg(u, Q_X) \geq (d(A,X)-\varepsilon)|Q_X| \quad \text{for all } X \in \mathcal{L}_0''.$$

By (21), we have $d(A,X) \geq 1-2\eta$ for $X \in \mathcal{L}_0$. We partition to-be-embedded $4\rho n$ root-2-paths into two groups, with $(4\rho - 5d)n$ paths in group 1 and $5dn$ paths in group 2. We embed the mid-points of the paths in group 1 into $N(u, Q_X), X \in \mathcal{L}_0''$, and the mid-points of the paths in group 2 into $N(u, P_X), X \in \mathcal{L}_0'$ for some $u \in A_0$ (note that P_X and Q_X are disjoint). This is possible because

$$\sum_{X \in \mathcal{L}_0''} \deg(u, Q_X) \geq |\mathcal{L}_0''|(1-2\eta-\varepsilon)|Q_X| \geq (1-\sqrt{\varepsilon})5\rho k(1-2\eta-\varepsilon)(1-2\sqrt{d}-\varepsilon)N > (4\rho-5d)n,$$

$$\sum_{X \in \mathcal{L}_0'} \deg(u, P_X) \geq |\mathcal{L}_0'|(1-2\eta-\varepsilon)|P_X| \geq (1-\sqrt{\varepsilon})5\rho k(1-2\eta-\varepsilon)2\sqrt{d}N > 5dn.$$

To finish the embedding, we choose an unoccupied (distinct) neighbor to be the leaf for each of the $(4\rho-5d)n$ vertices embedded in $Q_X, X \in \mathcal{L}_0''$. This is possible because each vertex in Q_X has degree at least $(1-5d)n$. Finally, we attach one leaf to each of the $5dn$ vertices embedded in $P_X, X \in \mathcal{L}_0'$. □

Let G_r' be the subgraph of G_r containing all regular pairs between $\mathcal{V}_1$ and $\mathcal{V}_2$ with density at least 2η. We claim that $e_{G_r'}(\mathcal{L}_1, \mathcal{V}_2)$ is small.

Claim 5.20. *$e_{G_r'}(\mathcal{L}_1, \mathcal{V}_2) < 16\rho_1 k^2$, where $\rho_1 = \rho^{1/3}$.*

Proof. We assume that there is a subset $\mathcal{L}_0 \subseteq \mathcal{L}_1$ of size $8\rho_1 k$ such that every $C \in \mathcal{L}_0$ has at least $8\rho_1 k$ G_r'-neighbors in $\mathcal{V}_2$ (neighbors in $\mathcal{V}_2$ with respect to G_r'). Otherwise $e_{G_r'}(\mathcal{L}_1, \mathcal{V}_2) < 8\rho_1 k|\mathcal{L}_1| + |\mathcal{V}_2|8\rho_1 k \leq 8\rho_1 k(2k)$, and we are done. By the definition of $\mathcal{L}_1$, the clusters in $\mathcal{L}_1$ must be large and located in different regular pairs in which the other ends are small clusters. The partner set $\mathcal{S}_0 = \{Y : \{X,Y\} \in \mathcal{M}_{in}, X \in \mathcal{L}_0\}$ of $\mathcal{L}_0$ is a subset of $\mathcal{S}_1$. Our goal is to derive that $e_{G_r}(\mathcal{S}_0, \mathcal{V}_2) \geq 16\rho k^2$, a contradiction to Claim 5.19.

Fix a cluster $C \in \mathcal{L}_0$. Let $\mathcal{M}_2$ be the set of $\{X,Y\} \in \mathcal{M}_{out}$ such that either X or Y is a G_r'-neighbor of C. Since $C \in \mathcal{O}$, Claim 5.11, Part 2 says that $\mathcal{M}_2$ contains at least $8\rho_1 k - 9\sqrt{d}k$ G_r'-neighbors of C. Let $\mathcal{M}_2'$ be the set of $\{X,Y\} \in \mathcal{M}_2$ such that $|d(C,X) - d(C,Y)| \leq \eta$. Since $C \in N(A) \cap \mathcal{O}$ and C is large, C and A may play the same roles of A and B, respectively. We then have $|\mathcal{M}_2'| \geq |\mathcal{M}_2| - \eta k$ by Claim 5.16. For any $\{X,Y\} \in \mathcal{M}_2'$, by the definition of G'_r, one of $d(C,X)$ and $d(C,Y)$ is at least 2η, consequently the other one is at least η. This implies that $\mathcal{M}'_2 \subseteq \mathcal{M}^2(C)$. By Claim 5.11, Part 3, all but one cluster in $V(\mathcal{M}'_2)$ are members of $\mathcal{O}$. We form a set $\tilde{N}(C) \subset \mathcal{L} \cap \mathcal{O}$ by picking one large cluster from each edge of $\mathcal{M}'_2$ unless it is not in $\mathcal{O}$. We have

$$|\tilde{N}(C)| \geq (8\rho_1 k - 9\sqrt{d}k)/2 - \eta k - 1 > 3\rho_1 k. \tag{25}$$

Let $\mathcal{N} = \cup_{C\in\mathcal{L}_0} \tilde{N}(C)$ (then $\mathcal{N} \subset \mathcal{L} \cap \mathcal{O}$). Define a bipartite graph H on $\mathcal{L}_0 \cup \mathcal{N}$ such that $C \in \mathcal{L}_0$ is adjacent to $D \in \mathcal{N}$ if and only if $D \in \tilde{N}(C)$. Let $\mathcal{N}_0$ be the set of $D \in \mathcal{N}$ such that $\deg_H(D) \geq 12\rho_1^2 k$. Since $|\mathcal{L}_0| = 8\rho_1 k$, (25) implies that

$$8\rho_1 k \; 3\rho_1 k = |\mathcal{L}_0|\, 3\rho_1 k \leq |E(H)| \leq |\mathcal{N}_0|\, 8\rho_1 k + |\mathcal{N}|\, 12\rho_1^2 k.$$

By using $|\mathcal{N}| \leq |\mathcal{V}_2| \leq (1+8\eta)k$, we obtain that $|\mathcal{N}_0| \geq 3(1-8\eta)\rho_1 k/2$. It suffices to show that $\deg_{G_r}(D, \mathcal{S}_0) \geq 11{\rho_1}^2 k$ for every $D \in \mathcal{N}_0$ because it gives the desired contradiction (by using $\rho_1 = \rho^{1/3}$)

$$e_{G_r}(\mathcal{N}_0, \mathcal{S}_0) > \frac{3}{2}\,(1-8\eta)\,\rho_1 k \; 11\rho_1^2 k > 16\rho k^2.$$

Fix a cluster $D \in \mathcal{N}_0$, and assume that $D \in \tilde{N}(C)$ for some $C \in \mathcal{L}_0$. Since $D, C \in \mathcal{L} \cap \mathcal{O}$ and $D \sim C$, we may let D and C play the role of A and B, respectively. By Claim 5.16, at most ηk pairs $\{X,Y\} \in \mathcal{M}_{in}$ satisfy $d(D,X) \geq \eta$ and $d(D,Y) = 0$. The definition of $\mathcal{N}_0$ implies that D has at least $12\rho_1^2 k$ G'_r-neighbors in $\mathcal{L}_0$. Since $\mathcal{S}_0$ is the partner set of $\mathcal{L}_0$ in $\mathcal{M}_{in}$, it follows that D has at least $12{\rho_1}^2 k - \eta k > 11{\rho_1}^2 k$ G_r-neighbors in $\mathcal{S}_0$. In other words, $\deg_{G_r}(D, \mathcal{S}_0) \geq 11{\rho_1}^2 k$. □

From Claim 5.19 and 5.20, we conclude that

$$e_{G'_r}(\mathcal{V}_1, \mathcal{V}_2) \leq e_{G_r}(\mathcal{S}_1, \mathcal{V}_2) + e_{G'_r}(\mathcal{L}_1, \mathcal{V}_2) < 16\rho k^2 + 16\rho_1 k^2 < 32\rho_1 k^2,$$

which implies that $e_{G''}(V_1, V_2) < (32\rho_1 + 2\eta)n^2$. Since $\alpha \geq 32\rho_1 + 2\eta$, it follows that G is in **EC2** with parameter α, contradiction. We now complete the proof of Theorem 3.3. □

6. The extremal cases

In this section we prove Proposition 3.1 and Theorem 3.2. The proof of Proposition 3.1 is straightforward, but a proof of Theorem 3.2 is far from trivial (at least from the point of our view). To prove it, we first define and handle a particular extremal case (denoted by **EC3**), in which the embedding of T mainly takes place in one partition set V_1 and then show that the assumption of Theorem 3.2, **EC2** actually implies **EC3**.

We first list a few facts to be used in both proofs.

Fact 6.1. *Let $0 < c < 1$ and $G = (V, E)$ be a graph of order n containing two disjoint vertex sets A and B. If $e(A, B) \geq (1-c)|A||B|$, then there exists a subset $B' \subseteq B$ such that*

$$|B'| \geq (1 - \sqrt{c})|B|, \;\; \delta(B', A) \geq (1 - \sqrt{c})|A|$$

Proof. Let $B' = \{u \in B : \deg(u, A) \geq (1 - \sqrt{c})|A|\}$ and $m = |B \setminus B'|$. Because

$$(1-c)|A||B| \leq e(A, B) \leq m\left(1 - \sqrt{c}\right)|A| + (|B| - m)\,|A|,$$

which implies that $m \leq \sqrt{c}|B|$. □

The naive greedy algorithm is the main tool of for embedding trees, as seen in Fact 1.1. Furthermore, given a tree T, if a graph G contains disjoint vertex sets A and B such that $\delta(A, B) \geq |T_{odd}|$, $\delta(B, A) \geq |T_{even}|$, then $T \subset G$. In particular, we can start our embedding by mapping any vertex $a \in A$ to any vertex $u \in T_{even}$ or any vertex $b \in B$ to any vertex $v \in T_{odd}$ (denoted by $a \to u$ and $b \to v$). The following fact gives a few variants of this embedding.

Fact 6.2. *Let $G = (V, E)$ be a graph with two disjoint vertex sets A and B. Then G contains a tree T if any of the following conditions holds.*

1. *$\delta(A, B), \delta(B, A) \geq \min\{|T_{even}|, |T_{odd}|\}$, and $\delta(A, V) \geq e(T)$.*
2. *Suppose that T has a vertex-partition $U_1 + U_2$ such that U_2 is independent but $U_1 \neq \emptyset$ is not necessarily independent. We have*

$$\min\{\delta(A, B), \delta(A, A), \delta(B, A)\} \geq |U_1|, \quad \text{and} \quad \delta(A, V) \geq e(T).$$

3. *Suppose that T has a vertex-partition $U_1 + U_2$ such that U_2 is independent. Let $\tilde{U}_2$ be a subset of U_2 containing all the nonleaf vertices of U_2. Fix $x \in U_1$, $a \in A$, $y \in \tilde{U}_2$, and $b \in B$. Then we can embed $T \to G$ such that either $x \to a$ or $y \to b$ if*

$$\delta(A, A), \delta(B, A) \geq |U_1|, \quad \delta(A, B) \geq |\tilde{U}_2|, \quad \text{and} \quad \delta(A, V) \geq e(T).$$

Proof. *Part 1.* Without loss of generality, assume that $|T_{even}| < |T_{odd}|$. Assume that $v(T) \geq 2$ otherwise $T \subset G$ is trivial. Applying Fact 5.14, we know that there are at least $|T_{odd}| - |T_{even}| + 1$ leaves in T_{odd}. We are thus able to put all the nonleaf vertices of T_{odd} into B, and all the vertices of T_{even} into A by the greedy algorithm. Since $\delta(A, V) \geq e(T)$, we can add the leaves of T_{odd} greedily.

Part 2. The proof is similar to Part 1, the only difference is that we need $\delta(A, A) \geq |U_1|$ when embedding U_1 because U_1 may not be independent.

Part 3. We first embed U_1 to A and $\tilde{U}_2$ to B by the greedy algorithm starting with $x \to a$ or $y \to b$. Since the vertices in $U_2 \backslash \tilde{U}_2$ are leaves, we can add them by the greedy algorithm. □

6.1. Extremal Case 1 (EC1)

In the proof below and later proofs, we often use the trivial fact that for any $T \subseteq S$, if $\deg(x, S) \geq |S| - s$, then $\deg(x, T) \geq |T| - s$.

Proof of Proposition 3.1. Given $0 < \sigma < 1$, let c be a real number such that $\sqrt[4]{c} + \sqrt{c} < (1 - \sqrt[4]{c})\sigma$ (in particular, $\sqrt{c} < \sigma$) and n_0 be the smallest integer n that satisfies

$$\left((1 - \sqrt[4]{c})\sigma - \sqrt[4]{c} - \sqrt{c}\right) n \geq 1 \tag{26}$$

Suppose that $n \geq n_0$. Let G be a $2n$-vertex graph such that $|L| \geq 2\sigma n$, where L is the set of vertices of degree at least n, and $V(G) = V_1 + V_2$ with $|V_1| = |V_2|$ and $d(V_1, V_2) \geq 1 - c$. Without loss of generality, we assume that $|V_1 \cap L| \geq \sigma n$. Since $e(V_1, V_2) > (1 - c)|V_1||V_2|$, we may apply Fact 6.1 to obtain $V_1' \subseteq V_1$ such that $|V_1'| \geq (1 - \sqrt{c})n$ and

$$\delta(V_1', V_2) \geq (1 - \sqrt{c})n. \tag{27}$$

Next we separate two cases based on the values of $t_e = |T_{even}|$ and $t_o = |T_{odd}|$.

Case a). $\min\{t_e, t_o\} \leq ((1 - \sqrt[4]{c})\sigma - \sqrt{c})\, n$.

Let $A = L \cap V_1'$. Since $|V_1'| \geq |V_1| - \sqrt{c}n$, we have $|A| \geq \sigma n - \sqrt{c}n$. Since $|V_2| = n$, (27) implies that $e(A, V_2) \geq (1 - \sqrt{c})|V_2||A|$. Applying Fact 6.1 again, we find $B \subseteq V_2$ such that $|B| \geq (1 - \sqrt[4]{c})n$ and

$$\delta(B, A) \geq (1 - \sqrt[4]{c})|A| \geq (1 - \sqrt[4]{c})(\sigma - \sqrt{c})n > ((1 - \sqrt[4]{c})\sigma - \sqrt{c})n.$$

On the other hand, (27) can be written as $\delta(V_1', V_2) \geq |V_2| - \sqrt{c}n$, which implies that

$$\delta(A, B) \geq |B| - \sqrt{c}n \geq (1 - \sqrt[4]{c} - \sqrt{c})n > ((1 - \sqrt[4]{c})\sigma - \sqrt{c})n$$

by using $\sigma \leq 1$. Since $\delta(A, B), \delta(B, A) \geq \min\{t_e, t_o\}$, we have $T \subset G$ from Fact 6.2 Part 1.

Case b). $\min\{t_e, t_o\} > ((1 - \sqrt[4]{c})\sigma - \sqrt{c})\, n$.

since $t_e + t_o = v(T) = n + 1$, we have $\max\{t_e, t_o\} < (1 - \sigma(1 - \sqrt[4]{c}) + \sqrt{c})n + 1$. By (27), $e(V_1', V_2) \geq (1 - \sqrt{c})|V_1'||V_2|$. We apply Fact 6.1 again to obtain a set $V_2' \subseteq V_2$ such that $|V_2'| \geq (1 - \sqrt[4]{c})n$ and

$$\delta(V_2', V_1') \geq (1 - \sqrt[4]{c})|V_1'| \geq (1 - \sqrt[4]{c})(1 - \sqrt{c})n > (1 - \sqrt{c} - \sqrt[4]{c})n.$$

We have $\delta(V_1', V_2') \geq |V_2'| - \sqrt{c}n \geq (1 - \sqrt[4]{c} - \sqrt{c})n$ from (27). The assumption (26) implies that

$$(1 - \sqrt[4]{c} - \sqrt{c})n > (1 - \sigma(1 - \sqrt[4]{c}) + \sqrt{c})n + 1,$$

and consequently $\delta(V_1', V_2'), \delta(V_2', V_1') \geq \max\{t_e, t_o\}$. We then apply the greedy algorithm to embed T into G. $\square$

6.2. Extremal Case 2 (EC2)

In this section, we prove Theorem 3.2 and also complete the proof of Theorem 1.9. Recall that a graph G is **EC2** with parameter α if there is a partition $V(G) = V_1 + V_2$ such that $|V_1| = |V_2| = n$, and $d(V_1, V_2) \le \alpha$.

We say that G is in the *Extremal Case 3* (**EC3**) with parameter θ if

- $V = V_1 + V_2, |V_1| = |V_2| = n$,
- There exists $A \subseteq V_1$ such that $|A| \ge n/2$, $\delta(A, V) \ge n$, and $\delta(A, V_1) \ge (1-\theta)n$.

Theorem 3.2 immediately follows from the next two lemmas. Note that we only need $\ell(G) \ge n/2 + 1$ for Lemma 6.3, which is much weaker than $\ell(G) \ge n$ provided by Theorem 3.2.

Lemma 6.3. *There exist $\theta_0 > 0$ and $n_0 \in \mathbf{N}$ such that for any $\theta \le \theta_0$ and $n \ge n_0$, if a $2n$-vertex graph G with $\ell(G) \ge n/2 + 1$ is in* **EC3** *with parameter θ, then $G \supset \mathcal{T}_n$.*

Lemma 6.4. *Let G be a graph on $2n$ vertices with $\ell(G) \ge n$. If G is in* **EC2** *with parameter α, then either $G \supset \mathcal{T}_n$ or G is in* **EC3** *with parameter $\theta \le 40\sqrt[4]{\alpha} + \sqrt{\alpha}$.*

We are ready to prove Theorem 1.9 now.

Proof of Theorem 1.9. Let c be given by Proposition 3.1 with $\sigma = 1/4$, and θ_0 be from Lemma 6.3. Given $0 < \beta < 1$, set $\alpha = \min\{c, \theta_0^2, \beta^2/9\}$.

Let $\varepsilon = \varepsilon(\alpha)$ be given by Theorem 3.3. Let $0 < \zeta \le 1/2$ such that $\zeta \le \varepsilon$ and $2\zeta \le \sqrt{\alpha} - 3\alpha$. Suppose that G is a $2n$-vertex graph for some large n such that $\ell(G) \ge (1-\zeta)n$ and $G \not\supset \mathcal{T}_n$. Since $\ell(G) \ge (1-\varepsilon)n$, Theorem 3.3 implies that G is in either of the two extreme cases with parameter α. Since $\zeta \le 1/2$, then $\ell(G) \ge n/2$. If G is in **EC1** with parameter α ($\le c$), then we can apply Proposition 3.1 with $\sigma = 1/4$ to obtain $G \supset \mathcal{T}_n$, a contradiction. This implies that G is in **EC2** with parameter α, namely, $V(G)$ can be evenly partitioned into V_1 and V_2 such that $d(V_1, V_2) \le \alpha$.

Let L be the set of vertices in G of degree at least n. We claim that $|V_i \cap L| < \frac{n}{2} + \sqrt{\alpha}n$ for $i = 1, 2$. In fact, let V_1' be the set of $x \in V_1$ such that $\deg(x, V_2) \ge \sqrt{\alpha}n$. Then $|V_1'| \le \sqrt{\alpha}n$ (otherwise $d(V_1, V_2) > \alpha$). Let $A' = (V_1 \cap L) \setminus V_1'$. If $|V_1 \cap L| \ge \frac{n}{2} + \sqrt{\alpha}n$, then $|A'| \ge n/2$. Consequently G is in **EC3** with parameter $\sqrt{\alpha}$ ($\le \theta_0$). Lemma 6.3 thus implies that $G \supset \mathcal{T}_n$, a contradiction. Since $|V_1 \cap L| + |V_2 \cap L| = |L| \ge (1-\zeta)n$, it follows that

$$\frac{n}{2} - \zeta n - \sqrt{\alpha}n < |V_i \cap L| < \frac{n}{2} + \sqrt{\alpha}n. \tag{28}$$

Let $A = V_1 \cap L$. We have

$$e(A, V_1) \ge |A|n - e(A_1, V_2) \ge |A|n - \alpha n^2.$$

After adding at most αn^2 edges, every $x \in A$ is adjacent to all other vertices in V_1. By (28), $G[V_1]$ becomes H_n after adding or removing at most $(\sqrt{\alpha} + \zeta)n^2$ more edges. Similarly we may change at most $\alpha n^2+(\sqrt{\alpha}+\zeta)n^2$ edges to transform $G[V_2]$ into H_n. After deleting αn^2 edges between V_1 and V_2, we finally transform G into $2H_n$. The total number of changed edges is at most

$$2(\alpha n^2+(\sqrt{\alpha}+\zeta)n^2)+\alpha n^2 \leq 3\sqrt{\alpha}n^2 \leq \beta n^2$$

by using $3\alpha+2\zeta \leq \sqrt{\alpha}$ and $3\sqrt{\alpha} \leq \beta$. □

6.2.1. Extremal Case 3

In this subsection we prove Lemma 6.3. Our proof is divided into two cases according to the number of the leaves in T.

Case 1: Embedding trees with at least $33\sqrt{\theta}n$ leaves.

The following three assertions are main ingredients in our proof. We postpone their proofs to the end.

Proposition 6.5. *Suppose that $0<\theta \leq (1-\sqrt{3}/2)^2$ and n is sufficiently large. Let G_1 be a graph of order n with a vertex set X such that $|X-n/2| \leq \theta n$ and $\delta(X,V(G_1)) \geq n-\theta n$. Then there exists $Y \subseteq V(G_1)\setminus X$ such that $\delta(X,Y) \geq |Y|-\theta n$, $\delta(Y,X) \geq |X|-\sqrt{\theta}n$, and $\delta(X,Y),\delta(Y,X) \geq \lceil n/2\rceil-\sqrt{\theta}n$.*

Definition 6.6. *Let T be a tree of size n such that $V(T)=U_1+U_2$.*

1). U_1+U_2 is called an ideal *partition if*

1. $|U_1| \leq |U_2|$,
2. *U_2 is independent,*
3. *U_1 contains at least $5\sqrt{\theta}n$ leaves, and U_2 contains at least $2\sqrt{\theta}n$ leaves.*

2). U_1+U_2 is called a near-ideal *partition if*

1. *$|U_1|=n/2+1$ and $|U_2|=n/2$ (so n is even),*
2. *U_2 is independent,*
3. *U_1 contains at least $5\sqrt{\theta}n$ leaves, and U_2 contains at least $2\sqrt{\theta}n$ leaves.*
4. *There exists a leaf $z\in U_1$ such that its parent $y\in U_2$ has degree 2.*

Lemma 6.7. *Let T be a tree with n edges and at least $33\sqrt{\theta}n$ leaves. Then either T has an ideal partition or a near-ideal partition.*

Lemma 6.8. *Suppose that $0\leq l<n$. Let T be a tree of size at most n, with a partition $V(T)=U_1+U_2$ such that U_1 contains at least $5l$ leaves and U_2 is independent. Let $\tilde{U}_2$ be*

a subset of U_2 *such that all vertices in* $U_2 \setminus \tilde{U}_2$ *are leaves* ($\tilde{U}_2$ *may contains leaves as well). If a graph* G *contains two disjoint vertex sets* X *and* Y *such that*

(i) $\delta(X, X), \delta(Y, X) \geq |X| - l,\ \ \delta(X, Y) \geq |Y| - l,\ \ \delta(X, Y) \geq |\tilde{U}_2|$,

(ii) $|X| \geq |U_1|,\ \delta(X, V(G)) \geq e(T)$,

then $T \subset G$. *Furthermore, for any* $x \in U_1$ *and* $a \in X$, *we can let* $x \to a$; *alternatively, for any leaf* $y \in \tilde{U}_2$ *and* $b \in Y$, *we can let* $y \to b$.

Proof of Lemma 6.3: T has at least $33\sqrt{\theta}n$ leaves. Let $G = (V, E)$ be a $2n$-vertex graph with $|L| \geq n/2 + 1$, where $L := \{x \in V : \deg(x) \geq n\}$. Suppose that there is a balanced partition $V(G) = V_1 \cup V_2$ such that V_1 contains a set $A \subseteq V_1 \cap L$ with $|A| \geq n/2$ and $\delta(A, V_1) \geq |V_1| - \theta n$. This implies that

$$|A| \geq \lceil n/2 \rceil, \quad \delta(A, V) \geq n, \quad \delta(A, A) \geq |A| - \theta n. \tag{29}$$

We further assume that $|A| \leq n/2 + \theta n$ (otherwise consider a subset of size $n/2 + \theta n$). Applying Proposition 6.5 with $G_1 = G[V_1]$ and $X = A$, we obtain a subset B_1 of $B := V_1 \setminus A$ such that

$$\delta(A, B_1) \geq |B_1| - \theta n,\ \delta(B_1, A) \geq |A| - \sqrt{\theta}n,\ \delta(A, B_1), d(B_1, A) \geq \lceil n/2 \rceil - \sqrt{\theta}n. \tag{30}$$

Let T a tree with n edges and at least $33\sqrt{\theta}n$ leaves. By Lemma 6.7, either T has an ideal partition or a near-ideal partition. First assume that T has an ideal partition $U_1 + U_2$. Then U_2 is independent, and U_1 contains at least $5\sqrt{\theta}n$ leaves. Since $|U_1| + |U_2| = n + 1$ and $|U_1| \leq |U_2|$, we have $|U_1| \leq \lceil n/2 \rceil \leq |A|$. Let W_2 be the set of all leaves in U_2. Then $|W_2| \geq 2\sqrt{\theta}n$. We also know that $|W_2| \geq |U_2| - |U_1| + 1$ from Fact 5.14, consequently $2|U_2| = (n + 1) + |U_2| - |U_1| \leq n + |W_2|$. Define $\tilde{U}_2 := U_2 \setminus W_2$. We have

$$|\tilde{U}_2| = |U_2| - |W_2| \leq (n + |W_2|)/2 - |W_2| = (n - |W_2|)/2 \leq n/2 - \sqrt{\theta}n, \tag{31}$$

Because of (29), (30) and (31), we can apply Lemma 6.8 with $l = \sqrt{\theta}n$, $X = A$ and $Y = B_1$ to embed T to G.

Second assume that T has a near-ideal partition. Since $|U_1| = n/2 + 1$, if $|A| \geq n/2 + 1$, then we can still apply Lemma 6.8 to embed T to G. Otherwise $|A| = n/2$. In this case we need $|L| \geq n/2 + 1$ and the assumption that some leaf $z \in U_1$ has its parent $y \in U_2$ of degree 2. Suppose that $x \in U_1$ is the parent of y.

We need to make some preparation. Let $B_2 = B \setminus B_1$. We know that $\delta(B_1, A)$ is near $n/2$ but $\delta(B_2, A)$ may be zero. Since $|V_1| = n$ and $\delta(A, V) \geq n$, then $\delta(A, V_2) \geq 1$; in particular, some vertex $v_2 \in V_2$ has at least one neighbor in A. If a vertex $v_1 \in B_2$ has no neighbor in A, then we may switch v_1 and v_2. Repeating this if necessary, we now assume that $\delta(B_2, A) \geq 1$.

Since $|L| \geq n/2 + 1$ and $|A| = n/2$, either $V_2 \cap L \neq \emptyset$ or $B \cap L \neq \emptyset$. First assume that $V_2 \cap L \neq \emptyset$ and pick a vertex $v_0 \in V_2 \cap L$. Since $\deg(v_0) \geq n$, v_0 has at least one neighbor in V_1.

Case I: v_0 is adjacent to a vertex $a \in A$.

Let $T' = T \setminus \{y, z\}$ and $G' = G \setminus \{v_0\}$. Then $V(T')$ has a partition $U_1' + U_2'$ with $U_1' = U_1 \setminus \{z\}$ and $U_2' = U_2 \setminus \{y\}$. We have $|U_1'| = n/2 = |A|$. Then (29), (30) and (31) still hold except that $\delta(A, V) \ge n$ is replaced with $\delta(A, V(G')) \ge n - 1 \ge e(T')$. We then apply Lemma 6.8 to embed T' to G' such that $x \to a$. Next map y to v_0, and finally add the leaf z by using $\deg(v_0) \ge n$.

Case II: v_0 is adjacent to a vertex $b \in B_1$.

Let $T' = T \setminus \{z\}$ and $G' = G \setminus \{v_0\}$. Then $V(T')$ has a partition $U_1' + U_2$ with $U_1' = U_1 \setminus \{z\}$. Since $|U_1'| = n/2 = |A|$ and $\delta(A, V(G')) \ge n - 1 \ge e(T')$, all the conditions of Lemma 6.8 are satisfied. Note that y is a leaf of T' but $y \in \tilde{U}_2$ because $\tilde{U}_2$ is the set of vertices in U_2 that are not leaves of T. We apply Lemma 6.8 to embed T' to G' such that $y \to b$, and then map z to v_0.

Case III: v_0 is adjacent to a vertex $b \in B_2$.

We have assumed that $\delta(B, A) \ge 1$. Let $a \in A$ be a neighbor of b. Let $T' = T \setminus \{y, z\}$ and $G' = G \setminus \{b, v_0\}$. Then $\delta(A, V(G')) \ge n - 2 \ge e(T')$. Since $b \notin B_1$, (30) still holds. We first apply Lemma 6.8 to embed T' to G' such that $x \to a$, and then map y to b and z to v_0.

Now assume that $B \cap L \neq \emptyset$ and pick a vertex $b \in B \cap L$. Then b has a neighbor $v_0 \in V_2$. We then follow Case II if $b \in B_1$ or Case III if $b \in B_2$. □

Proposition 6.5 follows from Fact 6.1 easily.

Proof of Proposition 6.5. Let $Y' = V(G_1) \setminus X$. Then $\delta(X, Y') \ge |Y'| - \theta n > (1 - 3\theta)|Y'|$ (because $n < 3|Y'|$). Hence $e(X, Y') > (1 - 3\theta)|X||Y'|$. By Fact 6.1, there is a subset $Y \subseteq Y'$ such that $\delta(Y, X) \ge (1 - \sqrt{3\theta})|X|$ and $|Y| \ge (1 - \sqrt{3\theta})|Y'|$. Since $|Y'| \ge n/2 - \theta n$, then $|Y| \ge (1 - \sqrt{3\theta})(n/2 - \theta n)$. Since $\delta(X, V(G_1)) \ge n - \theta n$, we have

$$\begin{aligned} \delta(X, Y) &\ge |Y| - \theta n \\ &\ge (1 - \sqrt{3\theta})(n/2 - \theta n) - \theta n \\ &> \left(\frac{1}{2} - \frac{\sqrt{3\theta}}{2} - 2\theta \right) n \\ &\ge \left\lceil \frac{n}{2} \right\rceil - \sqrt{\theta} n, \end{aligned}$$

where the last inequality holds when $\sqrt{3\theta}/2 + 2\theta < \sqrt{\theta}$ or $\theta < (1 - \frac{\sqrt{3}}{2})^2$, and n is sufficiently large. With $|X| \ge n/2 - \theta n$, the same computation shows that $\delta(Y, X) \ge (1 - \sqrt{3\theta})|X| \ge \lceil n/2 \rceil - \sqrt{\theta} n$. Finally $\delta(Y, X) \ge (1 - \sqrt{3\theta})|X| \ge |X| - \sqrt{\theta} n$ because $|X| \le n/2 + \theta n$ and $\theta < \frac{1}{\sqrt{3}} - \frac{1}{2}$. □

Let T be a rooted tree T and $x \in V(T)$. Recall that $T(x)$ is the maximal subtree of T containing x but not $p(x)$. Given $C \subset C(x)$, the subtree obtained from $T(x)$ by removing all $T(y), y \in C$ is called a *natural subtree rooted at x*. A natural subtree T' of T has the property that $T - E(T')$ is also a tree. The following simple fact on natural subtrees is needed for proving Lemma 6.7 and Claim 6.12.

Fact 6.9. *Let T be a rooted tree with $v(T)$ vertices and $w(T)$ leaves.*

*1. For any positive integer $k \le v(T)$, there is a natural subtree T' such that $\frac{k}{2} \le v(T) < k$ (called $[k/2, k]$-*subtree*).*

2. For any positive integer $k \le w(T)$, there always exists a natural subtree with m leaves such that $k/2 \le m < k$.

Proof. For $x \in V(T)$, write $t(x)$ for $v(T(x))$, where $T(x)$ is the maximal subtree containing x but not $p(x)$. In the partial order defined by T with $Rt(T)$ as the highest element, we find the lowest vertex x such that $t(x) \ge \frac{k}{2}$. Then $t(y) < \frac{k}{2}$ for every $y \in C(x)$. If $t(x) < k$, then $T(x)$ is the desired natural subtree. Otherwise, from $T(x)$, we repeat removing the subtree $T(y)$ for $y \in C(x)$ until the remaining subtree has order less than k. We know the size of this tree is at least $k/2$ because the last removed $y \in C(x)$ satisfies $t(y) < \frac{k}{2}$ and the subtree right before removing $T(y)$ has order at least k.

Part 2 follows similarly. □

Given a tree with a vertex-partition $U_1 + U_2$, *flipping* a vertex set S mean moving the vertices of S from one partition set to the other one.

The main procedure in the proof of Lemma 6.7 is to find a natural subtree T_0 rooted at r_0 such that both T_0 and $T - T_0$ have many leaves and then flip T_0 or $T_0 - r_0$ in the default partition (T_{even}, T_{odd}). In most cases, the resulting partition is an ideal partition. In other cases, we obtain a near-ideal partition.

Proof of Lemma 6.7. Without loss of generality, assume that $|T_{odd}| \ge |T_{even}|$. Let $g = |T_{odd}| - |T_{even}|$. Since $|T_{odd}| + |T_{even}| = n + 1$, then $g \ge 0$ has the same parity as $n + 1$. Denote sets of leaves in T_{even} and T_{odd} by W_e, W_o, respectively. Thus $|W_e| + |W_o| \ge 33\sqrt{\theta}n$. If $|W_e| \ge 5\sqrt{\theta}n$ and $|W_o| \ge 2\sqrt{\theta}n$, then $T_{even} + T_{odd}$ is an ideal partition, and we are done. Otherwise we have either $|W_o| < 2\sqrt{\theta}n$ or $|W_e| < 5\sqrt{\theta}n$.

(a) $|W_o| < 2\sqrt{\theta}n$.

Then $|W_e| > 31\sqrt{\theta}n$. We flip $2\sqrt{\theta}n - |W_o|$ vertices of W_e and their parents (not moving other vertices under the parents). Let U_1 and U_2 be the resulting sets obtained from T_{even} and T_{odd}, respectively. Clearly U_2 is independent and $|U_1| \le |T_{even}| \le |U_2|$. In addition, U_2 contains $2\sqrt{\theta}n$ leaves, and U_1 contains more than $5\sqrt{\theta}n$ leaves. Therefore $U_1 + U_2$ is an ideal partition.

(b) $|W_e| < 5\sqrt{\theta}n$.

Applying Fact 6.9, we find a natural subtree T_0 rooted r_0 that contains m leaves, where $11\sqrt{\theta}n \le m < 22\sqrt{\theta}n$ leaves. Then $T_1 := T - E(T_0)$ is also subtree, which contains at least $11\sqrt{\theta}n$ leaves. Since $|W_e| < 5\sqrt{\theta}n$, each of T_0 and T_1 contains at least $11\sqrt{\theta} - 5\sqrt{\theta} = 6\sqrt{\theta}n$ leaves (of T) in T_{odd}.

Set $g_0 = |V(T_0) \cap T_{odd}| - |V(T_0) \cap T_{even}|$. If $g_0 \geq g/2$ and $r_0 \in T_{even}$, then we flip T_0. Let U_2 and U_1 be the resulting sets generated from T_{even} and T_{odd}, respectively. Then

$$|U_1| - |U_2| = |T_{odd}| - |T_{even}| - 2(|V(T_0) \cap T_{odd}| - |V(T_0) \cap T_{even}|) = g - 2g_0 \leq 0,$$

and U_1 contains internal edges. In addition, U_1 contains at least $6\sqrt{\theta}n$ leaves (from T_1), and U_2 contains at least $6\sqrt{\theta}n$ leaves (from T_0). Therefore $U_1 + U_2$ is an ideal partition. If $g_0 \leq g/2$ and $r_0 \in T_{odd}$, then we can similarly get an ideal partition by flipping T_0.

If $g_0 \leq g/2 - 1$ and $r_0 \in T_{even}$, or $g_0 \geq g/2 + 1$ and $r_0 \in T_{odd}$, we can also obtain an ideal partition by flipping $T_0 \setminus \{r_0\}$.

If $g \equiv n + 1 \pmod 2$ is even, then these are all the cases and we are done. Now assume that g is odd (then n is even). The only remaining cases are (1). $g_0 = \frac{g-1}{2}$ and $r_0 \in T_{even}$, and (2). $g_0 = \frac{g+1}{2}$ and $r_0 \in T_{odd}$.

We flip T_0 in these cases. In Case (1), let U_2 and U_1 be the resulting sets generated from T_{even} and T_{odd}, respectively; while in Case (2), let U_1 and U_2 be the resulting sets generated from T_{even} and T_{odd}. Then $|U_1| = \frac{n}{2} + 1$, $|U_2| = \frac{n}{2}$, and U_2 is independent. Furthermore, U_1 and U_2 each contains more than $6\sqrt{\theta}n$ leaves. In order to claim that $U_1 + U_2$ is a near-ideal partition, we need to show that there exists $z \in U_1$ whose parent $y \in U_2$ has degree 2.

We claim that there exists a leaf $z \in U_1$, whose parent $p(z) \in U_2$ has exactly one nonleaf neighbor. Suppose we were in Case (1) before flipping. Let $W'_o = W_o \cap V(T_1)$. We know that $|W'_o| \geq 6\sqrt{\theta}n$. It suffices to show that there exists a leaf $z \in W'_o$ whose parent $p(z)$ has exactly one nonleaf neighbor. Suppose instead, for every $v \in W'_o$, its parent $p(v)$ has at least two nonleaf neighbors. Let $T'_1 = T_1 - W'_o$. Then two trees T'_1 and T_1 have the same number of leaves in T_{even}. We now use Fact 5.14 to find a lower bound for this number. Let $T^1_{even} = V(T_1) \cap T_{even}$ and $T^1_{odd} = V(T_1) \cap T_{odd}$. We know that $|T^1_{odd}| - |T^1_{even}| = g - g_0 = \frac{g+1}{2}$. The tree T'_1 has the bipartition $(T^1_{even} \cup \{r_0\}, T^1_{odd} - W_o)$. By Fact 5.14, the number of leaves of T'_1 in T_{even} is at least

$$(|T^1_{even}| + 1) - (|T^1_{odd}| - |W'_o|) + 1 \geq -\frac{g+1}{2} + 6\sqrt{\theta}n + 2 \geq 6\sqrt{\theta}n + 2 - \frac{2\sqrt{\theta}n + 1}{2} \geq 5\sqrt{\theta}n + 1.$$

All but at most one leaf of T_1 in T_{even} are leaves of T (the exception is r_0). Therefore T has at least $5\sqrt{\theta}n$ leaves in T_{even}, contradicting $|W_e| < 5\sqrt{\theta}n$. In Case (2), the arguments above show that there exists a leaf $z \in W_o \cap V(T_0)$ whose parent $p(z)$ has exactly one nonleaf neighbor and thus prove the claim.

Now consider the parent $y = p(z)$ of this leaf z. If y has another leaf neighbor z' (in U_1 since U_2 is independent), then we flip z, z' and y such that $U_1 + U_2$ becomes an ideal partition. We thus assume that $\deg(y) = 2$. Then $U_1 + U_2$ is a near-ideal partition. □

Given a vertex set C in a tree, let $p(C)$ denote the union of parents $p(x)$ for all $x \in C$.

Proof of Lemma 6.8. Let W_1 be the set of leaves in U_1 not including x. Then $|W_1| \geq 5l - 1$. Let $\hat{W}_1$ be the set of leaves at U_1 whose parent is located at U_2. We may only consider the case $|\hat{W}_1| \geq 4l$. Otherwise at least l leaves in U_1 have parents at U_1, and we move these leaves to U_2. Let U_1', U_2' be the resulting sets. Then $|U_1'| = |U_1| - l$ and U_2' is independent. Then $\tilde{U}_2$ is a subset of U_2' containing all the nonleaf vertices of U_2' and y. Conditions (i) and (ii) implies that

$$\delta(X, X), \delta(Y, X) \geq |X| - l \geq |U_1| - l = |U_1'|, \quad \delta(X, Y) \geq |\tilde{U}_2|, \quad \delta(X, V) \geq e(T).$$

Applying Fact 6.2, we can embed $T \to G$ with either $x \to a$ or $y \to b$.

Let $W_1' = \{v \in \hat{W}_1 : v$ is the unique leaf among the children of $p(v)\}$. First assume that $|W_1'| < 2l$. Let $W_1'' := \hat{W}_1 \setminus W_1'$ be the set of leaves in U_1, whose parents are in U_2, and each of these parents has at least two leaf children. Then $|W_1''| > 2l$ and $|p(W_1'')| \leq |W_1''|/2$. We flip $p(W_1'') \cup W_1''$. Let $U_1' + U_2'$ be the resulting sets. Then U_2' is independent and $|U_1'| = |U_1| - |W_1''| + |p(W_1'')| \leq |U_1| - l$ because $|W_1''| - |p(W_1'')| > l$. Let $\tilde{U}_2' = \tilde{U}_2 - p(W_1'')$. Then $|\tilde{U}_2'| < |\tilde{U}_2|$ and $y \in \tilde{U}_2'$ because $y \notin p(W_1'')$. We then apply Fact 6.2 to embed $T \to G$ such that either $x \to a$ or $y \to b$.

Now assume that $|W_1'| \geq 2l$. Since any two leaves in W_1' have two different parents, we have $|p(W_1')| = |W_1'|$. Let $U_1' = U_1 \setminus W_1'$. Since

$$\delta(X, X), \delta(Y, X) \geq |X| - l > |U_1| - 2l \geq |U_1'|, \quad \text{and} \quad \delta(X, Y) \geq |\tilde{U}_2|,$$

we can apply the greedy algorithm to embed $U_1 \setminus W_1'$ into X and $\tilde{U}_2$ into Y such that either $x \to a$ or $y \to b$. Note that we do not embed $W_2 := U_2 \setminus \tilde{U}_2$ at this moment. Let Y' be the set of images of $p(W_1')$. Since $|X| \geq |U_1| \geq |U_1 \setminus W_1'| + |W_1'|$, we can find a set $X' \subset X$ of $|W_1'|$ unoccupied vertices. Then $|X'| = |Y'| \geq 2l$. Since $\delta(X, Y) \geq |Y| - l$ and $\delta(Y, X) \geq |X| - l$, in the bipartite subgraph $G[Y', X']$, we have $\delta(Y', X') \geq |X'| - l \geq |X'|/2$, and $\delta(X', Y') \geq |Y'| - l \geq |Y'|/2$. The well-known marriage theorem thus provides a perfect matching from Y' to X', which in turn gives an embedding of W_1'. Finally, since W_2 is a set of leaves and $p(W_2)$ is embedded to X, we can add all the leaves in W_2 greedily. □

Case 2. Embedding trees with at most $33\sqrt{\theta}n$ leaves

In this case we need a lemma which generalizes the naive greedy algorithm and postpone its proof to the end.

Lemma 6.10. *Let $T = (U_1, U_2; E(T))$ be a tree with at most l leaves such that $|U_1|, |U_2| \geq 26l$. Let $G = (X, Y; E)$ be a bipartite graph satisfying*

1. $|X| \geq |U_1|, |Y| \geq |U_2|$,
2. $Y = Y_1 + Y_2$, $\delta(X, Y_1) \geq |Y_1| - l$, $\delta(Y_1, X) \geq |X| - l$,
3. $|Y_2| \leq l$, *and G contains $|Y_2|$ vertex-disjoint 2-paths, each of which consists of one vertex of Y_2 as mid-point and two vertices of X as end-points.*

Suppose that $z \in U_1$ and $a \in X$ such that a is not contained in the given 2-paths. Then T can be embedded to G such that $z \to a$.

Proof of Lemma 6.3: T **has at most** $33\sqrt{\theta}n$ **leaves.** Suppose that G is a graph in **EC3** with parameter θ (in this case we do not need $\ell(G) \geq n/2+1$). Then V_1 contains a set A such that (29) holds (though we do not need $\delta(A,A) \geq |A| - \theta n$ here). Without loss of generality, assuming $|A| = \lceil n/2 \rceil$. We apply Proposition 6.5 with $G_1 = G[V_1]$ and $X = A$ to obtain a subset B_1 of $V_1 \setminus A$ such that (30) holds. Let $\mathcal{F}$ be a maximum family of vertex-disjoint 2-paths with mid-points in $V \setminus (A \cup B_1)$ and end-points in A.

Let B_2 be the set of the mid-points from $\min\{|\mathcal{F}|, n-|A|-|B_1|\}$ paths of $\mathcal{F}$. Let $B = B_1 \cup B_2$ and $V_1' = A \cup B$. Then $n - \sqrt{\theta}n \leq |V_1'| \leq n$ because $|B| \geq |B_1| \geq \lceil n/2 \rceil - \sqrt{\theta}n$. We claim that $|V_1'| \geq n-1$ or equivalently $|B| \geq \lfloor n/2 \rfloor - 1$. Suppose to the contrary, that $|V_1| \leq n-2$. Then $|\mathcal{F}| < n - |A| - |B_1| \leq \sqrt{\theta}n$. Let A' be the set of the vertices of A that are *not* end-points of $\mathcal{F}$. Then $|A'| > n/2 - 2\sqrt{\theta}n$. For any vertex $v \in A'$, since $\deg(v) \geq n$ and $|V_1| \leq n-2$, we have $\deg(v, V_2') \geq n - (n-3) = 3$, where $V_2' := V \setminus V_1'$. The neighborhoods in V_2' of the vertices of A' must be disjoint, otherwise it yields a new 2-path which is vertex-disjoint from $\mathcal{F}$, contradicting the maximality of $\mathcal{F}$. But this implies that $3\left(\frac{n}{2} - 2\sqrt{\theta}n\right) > n + \sqrt{\theta}n \geq |V_2'|$, contradiction.

In summary, $G[V_1']$ satisfies Conditions 2 and 3 of Lemma 6.10 with $X = A$, $Y_1 = B_1$, $Y_2 = B_2$, and $l \geq \sqrt{\theta}n$. We also know that $A \subseteq L$, $A = \lceil n/2 \rceil$ and $\lfloor n/2 \rfloor - 1 \leq |B| \leq \lfloor n/2 \rfloor$.

Let T be a tree with n edges and at most $33\sqrt{\theta}n$ leaves. Without loss of generality, assume that $|T_{even}| \leq |T_{odd}|$. We also assume that $|T_{even}| > \lceil n/2 \rceil - \sqrt{\theta}n$ otherwise Fact 6.2 provides an embedding of T. First consider the case when $v(T) = n+1$ is odd. Then $|T_{even}| \leq n/2 = |A|$. By Proposition 5.14, T_{odd} has at least $|T_{odd}| - |T_{even}| + 1$ leaves. Let T_{odd}' be the set of non-leaf vertices in T_{odd}. Then $|T_{odd}'| \leq |T_{even}| - 1 \leq n/2 - 1 \leq |B|$. Let T' be the induced subtree of T on $T_{even} \cup T_{odd}'$. Then T' has at most $33\sqrt{\theta}n$ leaves with partition sizes $|T_{even}| > \lceil n/2 \rceil - \sqrt{\theta}n$ and $|T_{odd}'| \geq n/2 + 1 - 33\sqrt{\theta}n$. We have $|T_{even}|, |T_{odd}'| \geq 26(33\sqrt{\theta}n)$ because $\theta \ll 1$. With $l = 33\sqrt{\theta}n$, all the conditions of Lemma 6.10 are satisfied. We then apply Lemma 6.10 to embed T' into $G[V_1']$ such that $T_{even} \to A$ and $T_{odd}' \to B$. We complete the embedding of T by adding the leaves in T_{odd} greedily.

Now assume that $v(T) = n+1$ is even. If $|T_{even}| \leq (n-1)/2$, then we can proceed as before because $|T_{odd}'| \leq (n-3)/2 \leq |B|$. Otherwise $|T_{even}| = |T_{odd}| = (n+1)/2$. If either T_{even} or T_{even} has at least two leaves, then we can also embed as before because $|A| = (n+1)/2$ and $|B| \geq (n-3)/2$. Otherwise T has at most two leaves. Then T is a path (*e.g.*, following from Fact 6.11). The embedding of a path in any $2n$-vertex graph with $\ell(G) \geq n$ was proven in [3]. To make our proof self-contained, we show how to embed it under our current situation. Assume that $|V_1'| = n-1$ or $|B| = (n-3)/2$ otherwise we are done. Let A' be the set of the vertices of A that are *not* on the 2-paths covering B_2'. Fix a vertex $a \in A'$. Since $\deg(a, A') \geq |A'| - \theta n > 0$, we can find a neighbor $v \in A'$ of a. Let P be a path on $n-2$ vertices with leaves x and y. Then $|P_{even}| = (n-1)/2$ and $|P_{odd}| = (n-3)/2$. Let $A' = A \setminus \{v\}$; then $|A'| = (n-1)/2 = |P_{even}|$. All conditions of Lemma 5.14 hold with $U_1 = P_{even}$, $U_2 = P_{odd}$, $X = A'$, $Y = B$, and $l = \sqrt{\theta}n$. We apply Lemma 5.14 to embed P to $G[V_1' \setminus \{v\}]$ such that $x \to a$. Suppose that the other leaf y is mapped to $w \in A \setminus \{v\}$. We then extend P to a path on $n+1$ vertices by connecting a and v and adding a neighbor of v and a neighbor of w greedily.

We thus complete the proof of Lemma 6.3. $\square$

To prove Lemma 6.10, we need a few properties on trees with a small number of leaves. Given a vertex set S in a graph, write $N(S) = \bigcup_{x\in S} N(x)$. Given a 2-path uvw, we call it *S-2-path* if the mid-point $v \in S$, and call it *special 2-path* if all vertices in $N(\{u,w\})$ have degree at most two (which implies that $\deg(v) = 2$).

Proposition 6.11. *Let T be a tree with l leaves.*

1. $\sum_{x\in U^3}(\deg(x)-2) = l-2$, *where* $U^3 = \{x \in V(T) : \deg(x) \geq 3\}$. *In particular,* $|U^3| \leq l-2$.
2. $|N(S)| \leq 2|S| + l - 2$ *for any subset* $S \subset V(T)$.
3. *Suppose that T has the bipartition (U_1, U_2) such that $|U_1|, |U_2| \geq 26l$. Fix a vertex $z \in U_1$. Then T contains $5l$ special U_2-2-paths $P_1, \ldots, P_{5l}$ and $4l$ U_1-2-paths such that all these paths are vertex-disjoint and do not contain z.*

Proof. Define $U^i = \{x \in V(T) : \deg(x) = i\}$ for $i = 1, 2$ to make $U^1 \cup U^2 \cup U^3$ a partition of $V(T)$.

Part 1: $\sum_{x\in V(T)}(\deg(x)-2) = 2e(T) - 2v(T) = -2$. On the other hand,

$$-2 = \sum_{x\in V(T)}(\deg(x)-2) = -l + \sum_{x\in U^3}(\deg(x)-2),$$

which implies that $\sum_{x\in U^3}(\deg(x)-2) = l-2$.

Part 2: We partition S into S_1, S_2 and S_3 such that $S_i = \{x \in S : \deg(x) = 1\}$ for $i = 1, 2$, and $S_3 = \{x \in S : \deg(x) \geq 3\}$. Then

$$\begin{aligned} |N(S)| &\leq \sum_{x\in S}\deg(x) = |S_1| + 2|S_2| + \sum_{x\in S_3}\deg(x) \\ &= |S_1| + 2|S_2| + 2|S_3| + \sum_{x\in S}(\deg(x)-2) \\ &\leq 2|S| + (l-2), \quad \text{by Part 1.} \end{aligned}$$

Part 3: Let $U_j^i = U^i \cap U_j$ for $i = 1, 2, 3$ and $j = 1, 2$. Define the subset $U_2' = U_2^2 \setminus N(U_1^3 \cup \{z\})$, which consists of all $x \in U_2$ such that $\deg(x) = 2$, $z \notin N(x)$, and the two neighbors of x have degree at most two. For any subset $D \subseteq U_2'$, we can always find at least $|D|/3$ vertex-disjoint D-2-paths as follows. Suppose $x_1y_1z_1, \ldots, x_my_mz_m$ are m D-2-paths for some $m < |D|/3$. Then $|N(x_i) \cup N(z_i)| \leq 3$ for all i, and consequently there exists $y \in D$ such that $y \notin \bigcup_{i=1}^m N(x_i) \cup N(z_i)$. In other words, $y \notin \{y_1, \ldots, y_m\}$ and $N(y)$ is disjoint from $\{x_1, z_1, \ldots, x_m, z_m\}$. Hence y together with $N(y)$ (of size two) form a D-2-path that is vertex-disjoint from the existing U_2'-2-paths. We now give a bound for $|U_2'|$:

$$\begin{aligned}|U_2'| &\geq |U_2^2| - |N(U_1^3 \cup \{z\})| \\ &\geq |U_2| - |U_2^1| - |U_2^3| - 2(|U_1^3| + 1) - (l-2) \quad \text{by Part 2} \\ &\geq |U_2| - |U^1| - 2|U^3| - l \\ &\geq |U_2| - l - 2(l-2) - l \quad \text{by Part 1} \\ &> |U_2| - 4l.\end{aligned}$$

In order to find special U_2-2-paths, we consider $U_2'' = U_2' \setminus N^2(U_2^3)$, where $N^2(U_2^3) := N(N(U_2^3))$ is the set of the second-neighbors of U_2^3. Then every vertex $x \in U_2''$, its (two) neighbors, and its (at most two) second-neighbors all have degree at most two. Applying Part 2 twice,

$$|N^2(U_2^3)| \leq 2|N(U_2^3)| + (l-2) \leq 2(2|U_2^3| + l - 2) + l - 2 \leq 7(l-2),$$

and consequently $|U_2''| \geq |U_2'| - 7(l-2) \geq |U_2| - 11l$. Since $|U_2| \geq 26l$, we can find $|U_2''|/3 \geq (|U_2| - 11l)/3 \geq 5l$ vertex-disjoint U_2''-2-paths $P_1, \ldots, P_{5l}$. Furthermore, let

$$U_1' = (U_1^2 \setminus \{z\}) \setminus N(U_2^1) \quad \text{and} \quad \tilde{U}_1' = U_1' \setminus \cup_{i=1}^{5l} V(P_i).$$

Similar arguments as above gives that $|U_1'| \geq |U_1| - 4l$ and consequently $|\tilde{U}_1'| \geq |U_1'| - 2(5l) \geq |U_1| - 14\,l$. Since $|U_1| \geq 26l$, we can find $|\tilde{U}_1'|/3 \geq (|U_1| - 14\,l)/3 \geq 4l$ vertex-disjoint $\tilde{U}_1'$-2-paths $Q_1, \ldots, Q_{4l}$. Since the mid-points of $P_1, \ldots, P_{5l}$ have degree two and they already have two neighbors, they can not be the end-points of $Q_1, \ldots, Q_{4l}$. Then all $P_1, \ldots, P_{5l}, Q_1, \ldots, Q_{4l}$ are vertex-disjoint. In addition, our definition of U_1', U_2' guaranteed that z is not contained in any P_i or Q_i. □

Proof of Lemma 6.10. let $k := |Y_2| \leq l$ and denote the given Y_2-2-paths by $O_1, \ldots, O_k$. Applying Proposition 6.11, we find $4l + k$ special U_2-2-paths $P_1, \ldots, P_{4l+k}$ and $4l$ U_1-2-paths $Q_1, \ldots, Q_{4l}$ such that all the paths are vertex-disjoint and do not contain z. Fix the partial order of T with $z = Rt(T)$. For $i = 1, \ldots, k$, let t_i be the vertex closest to z among $V(P_{4l+i})$, and suppose that $s_i = p(t_i)$ and $r_i = p(s_i)$ are its parent and grand-parent, respectively. Note that s_i, r_i exist because $t_i, z \in U_1$ and $z \neq t_i$. Since each P_i is a special U_2-2-path, we have $\deg(s_i) \leq 2$, which implies $\deg(s_i) = 2$.

Let $F = T - \bigcup_{i=1}^{4l} E(P_i) \cup E(Q_i)$, that is, the forest obtained from T after removing the mid-points and the edges of P_i, Q_i for $i = 1, \ldots, 4l$. Then F has two partition sets F_e and F_o with $|F_e| = |U_1| - 4l$ and $|F_o| = |U_2| - 4l$, and F contains F_o-2-paths $P_{4l+1}, \ldots, P_{4l+k}$. We first embed $P_{4l+1}, \ldots, P_{4l+k}$ to $O_1, \ldots, O_k$, and then embed the remaining vertices of F into $X \cup Y_1$ by the greedy algorithm (following the partial order of T) such that $z \to a$, $F_e \to X$, and $F_o \to Y_1$.

For any vertex $x \in V(F) \setminus \{s_1, \ldots, s_k\}$, the embedding of x is guaranteed by

$$\delta(Y_1, X) \geq |X| - l \geq |U_1| - l > |F_e|, \quad \delta(X, Y_1) \geq |Y_1| - l \geq |Y| - 2l \geq |U_2| - 2l > |F_o|.$$

Assume that $u_i \in V(O_i)$ is the image of t_i. If $r_i \notin V(F)$, then s_i is a root of F and we simply map s_i to a neighbor of u_i. Otherwise $r_i \in V(F)$ is mapped to some vertex v_i by the greedy algorithm. Then we map s_i to an unoccupied common neighbor of u_i and v_i in Y_1. This is always possible because $|Y_1| - 2l \geq |U_2| - 3l > |F_o|$.

Since $|X| \geq |U_1| = |F_e| + 4l$, we can find a subset $\tilde{X} \subset X$ containing $4l$ unoccupied vertices. For $i = 1, \ldots, 4l$, let $p_i, q_i \in Y_1$ be the images of the end-vertices of Q_i. We form a bipartite graph $\tilde{B}$ on $\tilde{X}$ and $\tilde{Y} := \{y_1, \ldots, y_{4l}\}$ in which two vertices $x \in \tilde{X}$ and y_i are adjacent if and only if x is adjacent to both p_i and q_i. Since $\delta(Y_1, X) \geq |X| - l$, we have $\deg_{\tilde{B}}(y_i, \tilde{X}) \geq 4l - 2l = |\tilde{X}|/2$. On the other hand, $\delta(X, Y_1) \geq |Y_1| - l$ implies that $\deg_{\tilde{B}}(a, \tilde{Y}) \geq 4l - l > |\tilde{Y}|/2$. By the marriage theorem, there exists a perfect matching between $\tilde{X}$ and $\tilde{Y}$ in $\tilde{B}$. We accordingly add $4l$ $\tilde{X}$-2-paths to F. We repeat this process to add the missing vertices in U_2 and thus complete the embedding of T. □

6.2.2. Proof of Lemma 6.4

In this subsection we prove Lemma 6.4. Let G be a $2n$-vertex graph G in **EC2** with parameter α, *i.e.*, $V(G)$ can be partitioned into $V_1 \cup V_2$ such that $|V_1| = |V_2| = n$ and $d(V_1, V_2) \leq \alpha$. Let L be the set of vertices of degree at least n. Suppose that $|L| \geq n$. Assume that $\mathcal{T}_n \not\subset G$. Our goal is to show that G is in **EC3** with parameter $40\alpha^{\frac{1}{4}} + \sqrt{a}$. Now let $\alpha_1 = 40\alpha^{\frac{1}{4}}$.

Claim 6.12. *There is no vertex $v \in L$ such that* $\deg(v, V_1), \deg(v, V_2) \geq \alpha_1 n$.

Proof. Suppose instead, there exists $v_0 \in L$ such that $\deg(v, V_1), \deg(v, V_2) \geq \alpha_1 n$. Without loss of generality, assume that $v_0 \in V_1 \cap L$. Further assume that $\deg(v_0, V_1) \geq \frac{n}{2}$. The case when $\deg(v_0, V_2) \geq \frac{n}{2}$ is similar.

For $i = 1, 2$, let A_i be the set of $x \in V_i \cap L$ such that $\deg(x, V_j) \leq \sqrt{\alpha} n$ for $j \neq i$. Then $\delta(A_i, V_i) \geq (1 - \sqrt{\alpha})n$. Since $d(V_1, V_2) \leq \alpha$, we have $|A_i| \geq |V_i \cap L| - \sqrt{\alpha} n$. If $|V_i \cap L| \geq n/2 + \sqrt{\alpha} n$ for any i, then $|A_i| \geq n/2$ and consequently G is in **EC3** with parameter $\sqrt{\alpha}$. We may thus assume that $|V_i \cap L| < \frac{n}{2} + \sqrt{\alpha} n$ for $i = 1, 2$. Since $|L| \geq n$, this implies that $|V_i \cap L| > n/2 - \sqrt{\alpha} n$ for $i = 1, 2$. Consequently

$$\frac{n}{2} + \sqrt{\alpha} n > |V_i \cap L| \geq |A_i| \geq |V_i \cap L| - \sqrt{\alpha} n > \frac{n}{2} - 2\sqrt{\alpha} n.$$

Applying Proposition 6.5 with $\theta = 2\sqrt{\alpha}$, we obtain $B_i \subseteq V_i \setminus A_i$ such that

$$\delta(A_i, B_i) \geq \frac{n}{2} - 2\alpha^{\frac{1}{4}} n, \quad \delta(B_i, A_i) \geq \frac{n}{2} - 2\alpha^{\frac{1}{4}} n. \tag{32}$$

Let T be a tree of size n. We will show that $T \subset G$. If T has a partition $U_1 + U_2$ such that $|U_1| \leq \frac{n}{2} - 2\alpha^{\frac{1}{4}} n$ and U_2 is independent, then $T \subset G$ follows from Fact 6.2 because of (32) and

$$\delta(A_i, A_i) \geq |A_i| - \sqrt{\alpha} n \geq \frac{n}{2} - 3\sqrt{\alpha} n \geq \frac{n}{2} - 2\alpha^{\frac{1}{4}} n.$$

We thus assume that T has such a partition.

Applying Fact 6.9, we find an $[\frac{\alpha_1}{4}n, \frac{\alpha_1}{2}n]$-subtree T' rooted at r. We let r be the root of T. Note that $F = T - T'$ is a forest while $F \cup \{r\}$ spans a tree. We map r to v_0 and will embed $T' - r$ in V_2 and embed F in V_1. The embedding of $T' - r$ in V_2 is easy. From (32), we derive that $|A_2 \cup B_2| \geq n - 4\alpha^{\frac{1}{4}}n$. Together with $\deg(v_0, V_2) \geq \alpha_1 n$, this implies that $\deg(v_0, A_2 \cup B_2) > \alpha_1 n - 4\alpha^{\frac{1}{4}}n > \frac{\alpha_1}{2}n$. Since $\deg_{T'}(r) \leq \frac{\alpha_1}{2}n$, we are able to map the neighbors of r in T' to $A_2 \cup B_2$. Let $G_2 = G[A_2 \cup B_2]$. By (32), $\delta(G_2) \geq \frac{n}{2} - 2\alpha^{\frac{1}{4}}n > e(T')$ and we can embed $T' - r$ to G_2 by the greedy algorithm.

Since $r \to v_0$ and $v_0 \in L$, all the leaves that are adjacent to r can be added at the end. We only need to embed F' to V_1, where F' is the subforest obtained from F after removing all isolated vertices. We have $|Rt(F') \leq |V(F')|/2 \leq (n - \frac{\alpha_1}{4}n)/2 = \frac{n}{2} - \frac{\alpha_1}{8}n$. Since $\deg(v_0, V_1) \geq n/2$ and $|A_1 \cup B_1| \geq n - 4\alpha^{\frac{1}{4}}n$, then $\deg(v_0, A_1 \cup B_1) \geq \frac{n}{2} - 4\alpha^{\frac{1}{4}}n \geq \frac{n}{2} - \frac{\alpha_1}{8}n$ (here we need $\alpha_1 \geq 32\alpha^{\frac{1}{4}}$). Therefore we can map $Rt(F')$ to $N(v_0, A_1 \cup B_1)$. Let (X_1, Y_1) be the bipartition of F' such that the roots embedded to A_1 are in X_1, while the roots embedded to B_1 are in Y_1. If $\max\{|X_1|, |Y_1|\} \leq \frac{n}{2} - 2\alpha^{\frac{1}{4}}n$, then we can embed F' to $A_1 \cup B_1$ by the greedy algorithm. Otherwise, say, $|X_1| \geq \frac{n}{2} - 2\alpha^{\frac{1}{4}}n + 1$. Although $|Y_1| < n/2 - 2\alpha^{\frac{1}{4}}n$ in this case, we can not apply Fact 6.2 because Y_1 is supposed to be embedded to B_1 but $B_1 \not\subset L$. Suppose that $T' - r$ has the bipartition (X_2, Y_2) with $|X_2| \geq |Y_2|$. Let $U_1 = Y_1 \cup Y_2 \cup \{r\}$ and $U_2 = X_1 \cup X_2$. Then U_2 is independent. We claim that $|U_1| \leq \frac{n}{2} - 2\alpha^{\frac{1}{4}}n$, which contradicts our earlier assumption. In fact, since $|X_1| + |Y_1| = v(F') = n + 1 - v(T')$, we have

$$|Y_1| \leq n + 1 - v(T') - \Big(\frac{n}{2} - 2\alpha^{\frac{1}{4}}n + 1\Big) \leq \frac{n}{2} - v(T') + 2\alpha^{\frac{1}{4}}n.$$

Since $|Y_2| \leq (v(T') - 1)/2$, it follows that

$$\begin{aligned}|U_1| &\leq |Y_1| + \frac{v(T') - 1}{2} + 1 \leq \frac{n}{2} - \frac{v(T')}{2} + 2\alpha^{\frac{1}{4}}n + \frac{1}{2} \\ &\leq \frac{n}{2} - 3\alpha^{\frac{1}{4}}n + \frac{1}{2} < \frac{n}{2} - 2\alpha^{\frac{1}{4}}n\end{aligned}$$

because $\frac{v(T')}{2} \geq \frac{\alpha_1}{8}n \geq 5\alpha^{\frac{1}{4}}n$. □

We now finish the proof of Lemma 6.4. Let $L^1 = \{v \in L : \deg(v, V_1) > \alpha_1 n\}$ and $L^2 = \{v \in L : \deg(v, V_2) > \alpha_1 n\}$. Claim 6.12 implies that $L^1 \cap L^2 = \emptyset$. Since $\delta(L, V) \geq n$ and $2\alpha_1 n < n$, then $L_1 \cup L_2$ is a partition of L. Thus $\delta(L^1, V_1) \geq (1 - \alpha_1)n$, and $\delta(L^2, V_2) \geq (1 - \alpha_1)n$. Let $L^i_j = L^i \cap V_j$, for $1 \leq i, j \leq 2$. Since $d(V_1, V_2) \leq \alpha$ and $\alpha_1 \geq \sqrt{\alpha}$, we have $|L^1_2| < \sqrt{\alpha}n$ and $|L^2_1| < \sqrt{\alpha}n$. Let $V'_1 = V_1 \cup L^1_2 \setminus L^2_1$ and $V'_2 = V_2 \cup L^2_1 \setminus L^1_2$. Then $L^i \subseteq V'_i$ and $|V'_i| \geq n/2 - \sqrt{\alpha}n$ for $i = 1, 2$. We move at most $\sqrt{\alpha}n$ vertices of $V \setminus L$ between V'_1 and V'_2 such that $|V'_1| = |V'_2| = n$. Without loss of generality, assume that $|L^1| \geq n/2$. Since $\delta(L^1, V'_1) \geq (1 - \alpha_1)n - \sqrt{\alpha}n$, then G in **EC3** with parameter $\alpha_1 + \sqrt{\alpha}$, under new partition sets V'_1, V'_2 with $A = L^1$. □

We finally complete the proof of Theorem 3.2.

7. Concluding Remarks

- What is the smallest $m = m(n, n/2)$ such that *every* n-vertex graph with at least m vertices of degree at least $n/2$ contains *all* trees on n edges as subgraphs? We have shown that this number is between $n/2 - \sqrt{n} - 1$ and $n/2$. We feel that lower bound is closer to the truth. To verify it, because of the robustness of Theorem 3.3, it suffices to improve our proof on the extremal cases.
- The techniques proving the extremal cases can be applied to prove the $k \geq (1-\varepsilon)v(G)$ case of the Komlós-Sós Conjecture (exactly). Since the aim of this paper is to prove the $(n/2 - n/2 - n/2)$ Conjecture, we do not generalize our proof for this purpose.

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Note added in proof. After the first version of this paper was written and publicized in 2002, more work has been done on the Komlós-Sós Conjecture (Conjecture 1.4). Piguet and Stein [14] recently proved an approximate version of the conjecture. More recently Piguet and Hladky [12] and independently Cooley [6] combined the ideas from the present paper and [14] to prove Conjecture 1.4 for all $k = \Omega(n)$.

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A. Appendix

We prove Lemma 5.3 Part 3. The following corollary of Lemma 4.6 gives a sufficient condition for embedding a forest of small trees into two adjacent clusters with prescribed location of roots.

Corollary A.1. *Let $X, Y \in \mathcal{V}$ be two adjacent clusters containing subsets $P \subseteq X_1 \subseteq X$ and $Q \subseteq Y_1 \subseteq Y$. Let F be a forest consisting of trees of order at most εN. If $U_1 \cup U_2$ is a bipartition of $V(F)$ with $R_1 := Rt(F) \cap U_1$ and $R_2 := Rt(F) \cap U_2$ such that*

$$|U_1| \leq |X_1| - (\gamma + 3\varepsilon)N, |U_2| \leq |Y_1| - (\gamma + 3\varepsilon)N, |R_1| \leq |P| - 3\varepsilon N, |R_2| \leq |Q| - 3\varepsilon N, \quad (33)$$

then we can embed F with $U_1 \to X_1$, $U_2 \to Y_1$, $R_1 \to P$, and $R_2 \to Q$.

Proof. We describe an algorithm of embedding trees in F by applying Lemma 4.6 repeatedly while mapping as many non-root vertices as possible to $X \setminus P$ and $Y \setminus Q$. Assume that trees $T_1, \dots, T_{i-1}$ from F have been embedded such that $\bigcup_{j<i} V(T_j) \cap U_1 \to X_1$ and $\bigcup_{j<i} V(T_j) \cap U_2 \to Y_1$. Let X_1^*, Y_1^*, P^*, Q^* denote the sets of available vertices in X_1, Y_1, P, Q, respectively, at this moment. The assumption (33) implies that

$$|X_1^*|, |Y_1^*| \geq (\gamma + 3\varepsilon)N. \tag{34}$$

Suppose the next tree T_i in F has its root at U_1. Let $X_0 = X_1^* \setminus P$ if $|X_1^* \setminus P| \geq \gamma N$; otherwise $X_0 = X_1^*$. Similarly we define Y_0. In order to embed $T_i \to (P^*, X_0, Y_0)$ by Lemma 4.6. We need to verify that $|P^*| \geq 3\varepsilon N$ and $|X_0|, |Y_0| \geq \gamma N$. It is easy to see that, for example, $|X_0| \geq \gamma N$ follows from the definition of X_0 and (34). Since $|R_1| \leq |P| - 3\varepsilon N$, $|P^*| \leq 3\varepsilon N$ is only possible when P contains images of non-root vertices. This implies that $X_1 \setminus P$ has fewer γN vertices available before embedding T_{i-1}. Together with $|P^*| \leq 3\varepsilon N$, this implies that $|X_1^*| < (\gamma + 3\varepsilon)N$, contradiction. □

The proof of Lemma 5.3 Part 2 is somewhat technical. The main difficulty is that when embedding the first tree T_1 in F, we do not know the image u of the second root r_2, and can not purposely avoid $N(u, X)$ or $N(u, Y)$ as in the proof of Corollary A.1. It is natural to map $Rt(F)$ to the vertices that are typical to the sets of available vertices in X and Y. Nevertheless we may not be able to embed an *ordered* F to C, X, Y even if $F^o := F - Rt(F)$ has a bipartition similar to the one in Corollary A.1:

$$\begin{aligned} |U_1| &\leq |X| - (\gamma + 3\varepsilon)N, & |U_2| &\leq |Y| - (\gamma + 3\varepsilon)N, \\ |R_1| &\leq d(C, X)N - \varepsilon N - 3\varepsilon N, & |R_2| &\leq d(C, Y)N - \varepsilon N - 3\varepsilon N. \end{aligned} \tag{35}$$

Let us give an example. Construct a tripartite random graph on three sets C, X, Y of size N such that each edge appears with probability $1/3$ independently. Suppose that F consists of two εN-trees with roots r_1 and r_2. Accordingly $F^o = F - \{r_1, r_2\}$ is partitioned into F_1 and F_2, each of which consists of trees of order at most εN. Suppose that $v(F_1) = v(F_2) = (1 - 2\gamma)N$ and all trees in F_1, F_2 have ratio $1/2$ (for example, they are even paths). Furthermore, assume that $|Rt(F_1)| = \frac{N}{6}$ and $|Rt(F_2)| = (\frac{1}{4} - \gamma)2N$. Let (U_1, U_2) be a bipartition of F^o such that the roots of F_1 and F_2 are distributed evenly. Let $R_i = V(F^o) \cap U_i$. Then $|R_1| = |R_2| = (\frac{N}{6} + (\frac{1}{4} - \gamma)2N)/2 \leq \frac{N}{3} - 4\varepsilon N$, and $|U_1| = |U_2| = (1 - 2\gamma)N \leq N - (\gamma + 3\varepsilon)N$. Thus (35) holds. After embedding $r_1 \to C$ and $F_1 \to X \cup Y$, the sets X^* and Y^* of the remaining vertices are of size about $N/2$. However, for each vertex $u \in C$, we have $\deg(u, X^* \cup Y^*) = |X^* \cup Y^*|/3 \approx \frac{N}{3} < (\frac{1}{4} - \gamma)2N = |Rt(F_2)|$. Therefore we can not embed $Rt(F_2)$ for mapping r_2 to any vertex in C.

Let X, Y be adjacent clusters with $P \subseteq X$ and $Q \subseteq Y$, we write $F \to (P, Q; X, Y)$ if $F \to X \cup Y$ with $Rt(F) \to P \cup Q$. Let F be a forest of trees of size at most εN. Lemma 4.6 implies that $F \to (P, Q; X, Y)$ if $v(F)$ is about $|P| + |Q|$. This is sharp when F consists of isolated vertices. However, a larger F can be embedded when every tree in F has at least two vertices.

Lemma A.2. *Let $X, Y \in \mathcal{V}$ be two clusters with $X \sim Y$ containing subsets $P \subseteq X_1 \subseteq X$ and $Q \subseteq Y_1 \subseteq Y$. Suppose that F is a forest of trees of order between 2 and εN (inclusive). Then $F \to (P, Q; X_1, Y_1)$ if*

$$v(F) \le \min\Big\{2|P| + 2|Q| - 12\varepsilon N,\ \min\{|P|, |Q|\} + \min\{|X_1|, |Y_1|\} - (2\gamma + 7\varepsilon)N\Big\}.$$

Furthermore, let X_1^*, Y_1^* denote the sets of available vertices in X_1, Y_1 after F is embedded, and $X_1' = X_1 \setminus X_1^*$ and $Y_1' = Y_1 \setminus Y_1^*$. Then one of the following holds.

Case 1: $||X_1^*| - |Y_1^*|| \le \max\{||X_1| - |Y_1||, \varepsilon N\}$,

Case 2: $|X_1'| \ge |P| - 3\varepsilon N$ *and* $|Y_1'| \ge |P| - 4\varepsilon N$,

Case 3: $|Y_1'| \ge |Q| - 3\varepsilon N$ *and* $|X_1'| \ge |Q| - 4\varepsilon N$.

Proof. We show that there is a bipartition of $V(F)$ into U_1 and U_2, with $R_1 = Rt(F) \cap U_1$ and $R_2 = Rt(F) \cap U_2$ satisfying (33). Then $F \to (P, Q; X_1, Y_1)$ follows from Corollary A.1.

Without loss of generality, assume that $|X_1| \le |Y_1|$. Suppose that $F = \{T_1, \ldots, T_s\}$. For every $i \le s$, since $v(T_i) \le \varepsilon N$, then $|(T_i)_{even}|, |(T_i)_{odd}| \le \varepsilon N - 1$. By distributing the roots properly, we obtain a bipartition (U_1, U_2) of $T_1, \ldots, T_i$ such that $|U_1^i| \le |U_2^i| < |U_1^i| + \varepsilon N$. Let $R_1^i = Rt(F) \cap U_1^i$ and $R_2^i = Rt(F) \cap U_2^i$. If $|R_1^s| \le |P| - 3\varepsilon N$ and $|R_2^s| \le |Q| - 3\varepsilon N$, then set $U_1 := U_1^s$ and $U_2 := U_2^s$. We claim that (U_1, U_2) satisfies (33). We only need to verify that $|U_1| \le |X_1| - (\gamma + 3\varepsilon)N$ and $|U_2| \le |Y_1| - (\gamma + 3\varepsilon)N$. This, in fact, follows from

$$|U_1| \le |U_2| < |U_1| + \varepsilon N \tag{36}$$

and $|U_1| + |U_2| = v(F) \le 2\min\{|X_1|, |Y_1|\} - (2\gamma + 7\varepsilon)N$. Furthermore, $|Y_1| \ge |X_1|$ means that $|Y_1^*| + |Y_1'| \ge |X_1^*| + |X_1'|$. Since $|X_1'| = |U_1|$ and $|Y_1'| = |U_2|$, we have $|Y_1^*| - |X_1^*| \ge |U_1| - |U_2| > -\varepsilon N$ by (36). On the other hand, $|Y_1^*| - |X_1^*| = (|Y_1| - |X_1|) - (|U_2| - |U_1|) \le |Y_1| - |X_1|$ by (36). This implies that $||X_1^*| - |Y_1^*|| \le \max\{|Y_1| - |X_1|, \varepsilon N\}$, i.e., Case 1 holds.

Otherwise $|R_1^s| > |P| - 3\varepsilon N$ or $|R_2^s| > |Q| - 3\varepsilon N$. Suppose that $|R_1^s| > |P| - 3\varepsilon N$, which implies that $|R_1^i| = |P| - \varepsilon N$ for some $i < s$. We then add the remaining trees of F to U_1^i and U_2^i such that their roots are in U_2. We claim that the resulting sets U_1, U_2, R_1 and R_2 satisfy (33). First, it is impossible to have $|R_2| > |Q| - 3\varepsilon N$ because it implies that $v(F) \ge 2(|R_1| + |R_2|) > 2(|P| + |Q| - 6\varepsilon N)$, contradiction (here we use the assumption that every tree in F has at least 2 vertices). Second, we derive that $|U_2| \le |Y_1| - (\gamma + 3\varepsilon)N$ from $|U_1| + |U_2| = v(F) \le |Y_1| + |P| - (\gamma + 6\varepsilon)N$, and $|U_1| \ge |U_1^i| \ge |R_1^i| = |P| - 3\varepsilon N$. Similarly we obtain $|U_1| \le |X_1| - (\gamma + 3\varepsilon)N$ from $|U_1| + |U_2| = v(F) \le |X_1| + |P| - (\gamma + 7\varepsilon)N$, and $|U_2| \ge |U_2^i| \ge |U_1^i| - \varepsilon N \ge |P| - 4\varepsilon N$. Actually $|U_2^i| \ge |U_1^i|$. The weaker inequality $|U_2^i| \ge |U_1^i| - \varepsilon N$ matches its analogue $|U_1^i| \ge |U_2^i| - \varepsilon N$ in the case that $|R_2^s| > |Q| - 3\varepsilon N$. The second assertion that $|X_1'| \ge |P| - \varepsilon N$ and $|Y_1'| \ge |P| - 2\varepsilon N$ is immediate.

The case that $|R_2^s| > |Q| - 3\varepsilon N$ is similar. □

Proof of Lemma 5.3 Part 3. We map the roots of F to the vertices of C that are typical to the subsets of available vertices in X and Y, and apply Lemma A.2 to embed trees in F one by one. Suppose that $F = \{T_1, \ldots, T_s\}$ and $Rt(F) = \{r_1, \ldots, r_s\}$. Let $F_j = T_j - \{r_j\}$.

Suppose that $T_1, \ldots, T_i$ have been embedded. Let X^i and Y^i denote the sets of available vertices in X and Y, respectively (in particular, $X^0 = X$ and $Y^0 = Y$). We map the root r_{i+1} of T_{i+1} to a vertex $u_{i+1} \in C$ such that $|N(u_{i+1}, X^i)| \geq (d_x - \varepsilon)|X^i|$ and $|N(u_{i+1}, Y^i)| \geq (d_y - \varepsilon)|Y^i|$. We then attempt to embed the forest F_{i+1} by Lemma A.2 with $X_1 = X^i$, $Y_1 = Y^i$, $P = N(u_{i+1}, X^i)$, and $Q = N(u_{i+1}, Y^i)$. Suppose to the contrary, that

$$v(F_{i+1}) > \min\Big\{2|P| + 2|Q| - 12\varepsilon N, \min\{|P|, |Q|\} + \min\{|X^i|, |Y^i|\} - (2\gamma + 7\varepsilon)N\Big\}. \quad (37)$$

Our goal is to derive that $|F| > (d_x + d_y + \lambda - 2\gamma - 13\varepsilon)N$, a contradiction.

As a preparation, we first show that $X' := X - X^i$ and $Y' := Y - Y^i$ satisfy one of the following conditions. Note that $|X'| + |Y'| = \sum_{j=1}^{i} v(F_j)$.

Case a: $||X'| - |Y'|| < \varepsilon N$,

Case b: $|X'| \geq d_x N - 4\varepsilon N$ and $|Y'| \geq d_x N - 5\varepsilon N$,

Case c: $|Y'| \geq d_y N - 4\varepsilon N$ and $|X'| \geq d_y N - 5\varepsilon N$.

This, in fact, follows from the second assertion of Lemma A.2, which says that one of the Cases 1, 2, and 3 must occur after the embedding of T_j for each $j \leq i$. First assume that Lemma A.2 Case 1 holds after embedding all T_j. Since $|X^0| = |Y^0| = N$, this implies that $||X^j| - |Y^j|| \leq \varepsilon N$ for all j. Consequently $||X'| - |Y'|| < \varepsilon N$. Second assume that Lemma A.2 Case 2 holds for the first time after embedding T_j for some $1 \leq j \leq i$. The statement of Case 2 implies that

$$|V(F_j) \cap X| \geq |N(u_j, X^{j-1})| - 3\varepsilon N \quad \text{and} |V(F_j) \cap Y| \geq |N(u_j, X^{j-1})| - 4\varepsilon N.$$

We also have $||X^{j-1}| - |Y^{j-1}|| \leq \varepsilon N$ because Lemma A.2 Case 1 holds before embedding T_j. Consequently

$$|X'| \geq |X| - |X^{j-1}| + |V(F_j) \cap X| \geq |X| - |X^{j-1}| + |N(u, X^{j-1})| - 3\varepsilon N \geq d_x N - 4\varepsilon N,$$

where the last inequality follows from the choice of u_j. On the other hand,

$$|Y'| \geq |Y| - |Y^{j-1}| + |V(F_j) \cap Y| \geq |Y| - |X^{j-1}| - \varepsilon N + |N(u_j, X^{j-1})| - 4\varepsilon N \geq d_x N - 5\varepsilon N.$$

The same arguments lead to Case c when Lemma A.2 Case 3 applies after embedding some T_j with $1 \leq j \leq i$.

Without loss of generality, assume that $|X^i| \leq |Y^i|$. For convenience, let $x_1 = |X^i|, y_1 = |Y^i|$, $x_0 = |X'|$ and $y_0 = |Y'|$ (so $x_0 + x_1 = y_0 + y_1 = N$). Recall that $P = N(u_{i+1}, X^i)$ and $Q = N(u_{i+1}, Y^i)$. We know that $|P| \geq d_x x_1 - \varepsilon N$ and $|Q| \geq d_y y_1 - \varepsilon N$. The assumption (37) implies the following two possible cases.

Case I: $v(F_{i+1}) > \min\{|P|, |Q|\} + \min\{|X^i|, |Y^i|\} - (2\gamma + 7\varepsilon)N$.

Since $|X^i| \le |Y^i|$, we have only two possibilities:

$$v(F_{i+1}) > |P| + |X^i| - (2\gamma + 7\varepsilon)N \ge d_x x_1 + x_1 - (2\gamma + 8\varepsilon)N \tag{38}$$

or

$$v(F_{i+1}) > |Q| + |X^i| - (2\gamma + 7\varepsilon)N \ge d_y y_1 + x_1 - (2\gamma + 8\varepsilon)N \tag{39}$$

Since $||F|| \ge x_0 + y_0 + v(F_{i+1})$ and $d_y y_1 + y_0 \ge d_y N$, (39) implies that $||F|| > N + d_y N - (2\gamma+8\varepsilon)N$. Since $d_x \le 1-\lambda$, we have $||F|| > (d_x + d_y + \lambda)N - (2\gamma+8\varepsilon)N$, a contradiction. When (38) holds, we have $||F|| > x_0 + y_0 + x_1 + x_1 d_x - (2\gamma + 8\varepsilon)N$. Now we proceed under three possible cases we have prepared.

- Under Case a, $y_0 \ge x_0 - \varepsilon N$, we have $||F|| > N + d_x N - (2\gamma + 9\varepsilon)N$.
- Under Case b, $y_0 \ge d_x N - 5\varepsilon N$, we have $||F|| > N + d_x N - (2\gamma + 13\varepsilon)N$.
- Under Case c, $y_0 \ge d_y N - 3\varepsilon N$, we have $||F|| > N + d_y N - (2\gamma + 11\varepsilon)N$.

All these cases lead to the desired contradiction, $||F|| > (d_x + d_y + \lambda)N - (2\gamma + 13\varepsilon)N$, because $d_x \le d_y \le 1 - \lambda$.

Case II: $v(F_{i+1}) > 2|P| + 2|Q| - 12\varepsilon N$. Since $|P| \ge d_x x_1 - \varepsilon N$ and $|Q| \ge d_y y_1 - \varepsilon N$,

$$\begin{aligned}||F|| &\ge x_0 + y_0 + 2(d_x x_1 + d_y y_1 - 2\varepsilon N) - 12\varepsilon N\\ &= d_x N + d_y N + (x_0 + d_x N - 2d_x x_0) + (y_0 + d_y N - 2d_y y_0) - 16\varepsilon N.\end{aligned}$$

Since $2\gamma > 3\varepsilon$, it suffices to show that $(x_0 + d_x N - 2d_x x_0) + (y_0 + d_y N - 2d_y y_0) \ge \lambda N$. Observe that all

$$x_0 - d_x x_0, \quad d_x N - d_x x_0, \quad y_0 - d_y y_0, \quad d_y N - d_y y_0$$

are non-negative numbers. When $x_0 \ge N/2$, we have $x_0 + d_x N - 2d_x x_0 \ge x_0 - d_x x_0 \ge \lambda N/2$ (by using $d_x \le 1 - \lambda$). Similarly when $y_0 \ge N/2$, we have $y_0 + d_y N - 2d_y y_0 \ge \lambda N/2$. If both $x_0 \ge N/2$ and $y_0 \ge N/2$, then

$$(x_0 + d_x N - 2d_x x_0) + (y_0 + d_y N - 2d_y y_0) \ge \lambda N/2 + \lambda N/2 = \lambda N,$$

as desired. Otherwise assume that $x_0 < N/2$. It is easy to see that $f(d_x) := x_0 + d_x N - 2d_x x_0$ is an increasing function of d_x. Since $d_x \ge \lambda$, this implies that $f(d_x) \ge x_0 + \lambda N - 2\lambda x_0 \ge \lambda N$ (since $\lambda \le 1/2$). The case $y_0 < N/2$ is the same. □