

Non-Uniform Turán-Type problems

Dhruv Mubayi* Yi Zhao †

Department of Mathematics, Statistics, and Computer Science
University of Illinois at Chicago
Chicago, IL 60607

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Abstract

Given positive integers n, k, t , with $2 \leq k \leq n$, and $t < 2^k$, let $m(n, k, t)$ be the minimum size of a family \mathcal{F} of (nonempty distinct) subsets of $[n]$ such that every k -subset of $[n]$ contains at least t members of \mathcal{F} , and every $(k-1)$ -subset of $[n]$ contains at most $t-1$ members of \mathcal{F} . For fixed k and t , we determine the order of magnitude of $m(n, k, t)$. We also consider related Turán numbers $T_{\geq r}(n, k, t)$ and $T_r(n, k, t)$, where $T_{\geq r}(n, k, t)$ ($T_r(n, k, t)$) denotes the minimum size of a family $\mathcal{F} \subset \binom{[n]}{\geq r}$ ($\mathcal{F} \subset \binom{[n]}{r}$) such that every k -subset of $[n]$ contains at least t members of \mathcal{F} . We prove that $T_{\geq r}(n, k, t) = (1 + o(1))T_r(n, k, t)$ for fixed r, k, t with $t \leq \binom{k}{r}$ and $n \rightarrow \infty$.

1 Introduction

Given positive integers n, k, t , with $2 \leq k \leq n$ and $t < 2^k$. We call a family $\mathcal{F} \subset 2^{[n]} \setminus \emptyset$ a (k, t) -system if every k -subset of $[n]$ contains at least t sets from \mathcal{F} , and every $(k-1)$ -subset of $[n]$ contains at most $t-1$ sets from \mathcal{F} . Analogously, given integers n, k, t, r , with $1 \leq r \leq k \leq n$ and $0 \leq t < 2^k$, a *Turán- $\geq r$* (n, k, t)-system (Turán- r (n, k, t)-system) is a family $\mathcal{F} \subset \binom{[n]}{\geq r}$ ($\mathcal{F} \subset \binom{[n]}{r}$) so that every k -subset of $[n]$ contains at least t members of \mathcal{F} . We denote by $m(n, k, t)$ the minimum size of a (k, t) -system, and by $T_{\geq r}(n, k, t)$ ($T_r(n, k, t)$) the minimum size of a *Turán- $\geq r$* (n, k, t)-system (Turán- r (n, k, t)-system).

Computer scientists introduced and studied $m(n, k, t)$ (see [4, 5, 7] for its history and applications). [7] proves that $m(n, k, t) = \Theta(n^{k-1})$ for $1 < t < k$ and $m(n, 3, 2) = \binom{n-1}{2} + 1$, and [4] proves that for fixed k , $m(n, k, 2) = (1 + o(1))T_{k-1}(n, k, 2)$. $T_r(n, k, t)$ (especially $T_r(n, k, 1)$) is well-known and sometimes called the generalized Turán number, though its nonuniform version $T_{\geq r}(n, k, t)$ appears not to have been studied before. Note that when $T_r(n, k, t) = \Omega(n^r)$ (for fixed $t, r < k$),

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the asymptotics of $T_r(n, k, t)$ are not known for any $r \geq 3$ and $t \geq 1$ (see [1] for an introduction to this problem, and [3, 6] for surveys in the case $t = 1$).

In this note, we first study $m(n, k, t)$ for all $1 \leq t < 2^k$, determining its order of magnitude for fixed k, t .

Theorem 1. *Let $2 \leq k \leq n$, $2 \leq t < 2^k$, and $\binom{k}{\leq j-1} < t \leq \binom{k}{\leq j}$. Then there exists a constant c , depending only on k and t , such that*

$$c \binom{n}{k-j} \leq m(n, k, t) \leq \binom{n}{k-j} \quad \text{for } t \leq \binom{k-1}{j} \quad (*)$$

$$\frac{1}{c} \binom{n}{j} \leq m(n, k, t) \leq \binom{n}{\leq j} \quad \text{for } t > \binom{k-1}{j}.$$

Remark. When $j = 1$, (*) yields $m(n, k, t) = \Theta(n^{k-1})$ for $2 \leq t \leq k-1$, a result from [7].

We can obtain the exact value of $m(n, k, t)$ for some choices of t . For $t = 1, k$, and $2^k - 1$, it is trivial to see that $m(n, k, t)$ is equal to $\binom{n}{k}$, n , and $\binom{n}{\leq k}$, respectively. We claim that $m(n, k, 2^k - 2) = \binom{n}{\leq k-1}$. To see this, we call a (k, t) -system \mathcal{H} *minimal* if $\sum_{S \in \mathcal{H}} |S| \leq \sum_{S \in \mathcal{H}'} |S|$ for every (k, t) -system \mathcal{H}' . If \mathcal{F} is a minimal (k, t) -system for $t = 2^k - 2 \geq 2^{k-1}$, and $A \in \mathcal{F}$, then $2^A \setminus \emptyset \subset \mathcal{F}$, since replacing A by $B \subset A$ for some $B \notin \mathcal{F}$ creates another (k, t) -system that contradicts the minimality of \mathcal{F} . Consequently $\mathcal{F}_k = \emptyset$, because $S \in \mathcal{F}_k$ now implies that $\mathcal{F}_k \setminus S$ is a (k, t) -system. Since $t = 2^k - 2$, we must have $\mathcal{F} = \binom{[n]}{\leq k-1}$.

Before proceeding our upcoming Theorem 3 which relates $T_{\geq r}(n, k, t)$ and $T_r(n, k, t)$, we make the following observation.

Observation 2. *Let $1 \leq r \leq k$ and $0 \leq t < \sum_{i=r}^k \binom{k}{i}$. Let j be the unique integer satisfying $\sum_{i=r}^{j-1} \binom{k}{i} \leq t < \sum_{i=r}^j \binom{k}{i}$ and let $t_0 = t - \sum_{i=r}^{j-1} \binom{k}{i} \geq 0$. If \mathcal{F} is a Turán- $j(n, k, t_0)$ -system, then $\mathcal{F}' = \bigcup_{i=r}^{j-1} \binom{[n]}{i} \cup \mathcal{F}$ is a Turán- $\geq r(n, k, t)$ -system. This implies that $T_{\geq r}(n, k, t) \leq \sum_{i=r}^{j-1} \binom{n}{i} + T_j(n, k, t_0)$.*

Theorem 3. *Let r, k, t, j, t_0 be fixed as in Observation 2.*

1. *If $t_0 = 0$, then $T_{\geq r}(n, k, t) = \sum_{i=r}^{j-1} \binom{n}{i}$.*
2. *If $t_0 \geq 1$, then $T_{\geq r}(n, k, t) = (1 + o(1)) \left(\sum_{i=r}^{j-1} \binom{n}{i} + T_j(n, k, t_0) \right)$.*

Conjecture 4. *Given r, k, t, j, t_0 as in Observation 2, $T_{\geq r}(n, k, t) = \sum_{i=r}^{j-1} \binom{n}{i} + T_j(n, k, t_0)$.*

Most of our notations are standard: Given a set X and an integer a , let $\binom{X}{a} = \{S \subset X : |S| = a\}$, $\binom{X}{\leq a} = \{S \subset X : 1 \leq |S| \leq a\}$, $\binom{X}{\geq a} = \{S \subset X : |S| \geq a\}$, and $2^X = \{S : S \subset X\}$. For $\mathcal{F} \subset 2^{[n]}$, let $\mathcal{F}_t = \mathcal{F} \cap \binom{[n]}{t}$ and $\overline{\mathcal{F}}_t = \binom{[n]}{t} \setminus \mathcal{F}_t$. Let $\mathcal{F}_{\leq t} = \bigcup_{i \leq t} \mathcal{F}_i$ and $\mathcal{F}_{\geq t} = \bigcup_{i \geq t} \mathcal{F}_i$. Write $\mathcal{F}(X)$ for $\mathcal{F} \cap 2^X$. An r -graph on X is a (hyper)graph $\mathcal{F} \subset \binom{X}{r}$.

2 Proofs

Proof of Theorem 1. The Theorem follows easily from the following four statements.

- (1) If $\binom{k-1}{j-1} < t \leq \binom{k}{j}$, then $m(n, k, t) \leq \binom{n}{k-j}$.
- (2) If $\binom{k-1}{j} < t \leq \binom{k}{\leq j}$, then $m(n, k, t) \leq \binom{n}{\leq j}$.
- (3) If $t > \binom{k}{\leq j-1}$, then $m(n, k, t) \geq \binom{n}{j} / \binom{k}{j}$.
- (4) If $\binom{k}{\leq j-1} < t \leq \binom{k-1}{j}$, then $m(n, k, t) \geq c \binom{n}{k-j}$, where c depends only on k and j .

The proofs of (1) and (3) are straightforward, so we only prove (2) and (4).

Proof of (2). Consider the smallest $i' \in [1, j]$ and the largest $i \in [1, j]$ such that $1 + \sum_{\ell=i'}^j \binom{k-1}{\ell} \leq t \leq \sum_{\ell=i}^j \binom{k}{\ell}$. Such i, i' exist since $\binom{k-1}{j} < t \leq \binom{k}{\leq j}$. We first show that $i' \leq i$. This is trivial for $i = j$, so assume that $i < j$. The choice of i implies that

$$t > \sum_{\ell=i+1}^j \binom{k}{\ell} = \binom{k}{i+1} + \sum_{\ell=i+2}^j \binom{k}{\ell} \geq \left[\binom{k-1}{i} + \binom{k-1}{i+1} \right] + \sum_{\ell=i+2}^j \binom{k-1}{\ell} + 1.$$

Since this is equal to $\sum_{\ell=i}^j \binom{k-1}{\ell} + 1$, the choice of i' implies that $i' \leq i$. Now let $\mathcal{F} = \cup_{\ell=i'}^j \binom{[n]}{\ell}$. Every k -set of $[n]$ has $\sum_{\ell=i}^j \binom{k}{\ell} \geq t$ members of \mathcal{F} ; every $(k-1)$ -set of $[n]$ has $\sum_{\ell=i}^j \binom{k-1}{\ell} \leq \sum_{\ell=i'}^j \binom{k-1}{\ell} \leq t-1$ members of \mathcal{F} . Consequently, $m(n, k, t) \leq |\mathcal{F}| \leq \binom{n}{\leq j}$. \square

Proof of (4). First, the assumption $\binom{k}{\leq j-1} < \binom{k-1}{j}$ implies that $j < k-j$. Let \mathcal{F} be a (k, t) -system. Let $K_{k-1}^{(i)}$ denote the complete i -graph of order $k-1$. Then $K_{k-1}^{(i)} \not\subseteq \mathcal{F}$ for all $i \in [j, k-1-j]$, otherwise we obtain a $(k-1)$ -set which contains $\binom{k-1}{i} \geq \binom{k-1}{j} \geq t$ members of \mathcal{F} , a contradiction. Recall that the Ramsey number $R^{(i)}(s, t)$ is the smallest N such that every i -graph on N vertices contains a copy of either $K_s^{(i)}$ or $\overline{K_t^{(i)}}$. By Ramsey's theorem, $R^{(i)}(s, t)$ is finite. Define $m_{k-2j+1} = k$, and $m_\ell = R^{(k-j-\ell)}(k-1, m_{\ell+1})$ recursively for $\ell = k-2j, k-2j-1, \dots, 2, 1$.

We claim that every m_1 -set of $[n]$ contains at least one member of $\mathcal{F}_{\geq k-j}$. Indeed, consider an m_1 -set S_1 . Because $K_{k-1}^{(k-j-1)} \not\subseteq \mathcal{F}$, the definition of m_1 implies that there exists a m_2 -subset $S_2 \subseteq S_1$ with all of its $(k-j-1)$ -subsets absent from \mathcal{F} . Repeating this analysis, we find a sequence of subsets $S_3 \supseteq \dots \supseteq S_{k-2j+1} = S$ of sizes $m_3 > \dots > m_{k-2j+1} = k$, respectively. The k -set S thus contains no members of \mathcal{F} of size $k-j-1, \dots, j$. On the other hand, the k -set S must contain at least $t > \binom{k}{\leq j-1}$ members of \mathcal{F} , thus at least one member of $\mathcal{F}_{\geq j}$. Hence S contains a member of $\mathcal{F}_{\geq k-j}$. By an easy averaging argument, we obtain $|\mathcal{F}| \geq \binom{n}{k-j} / \binom{m_1}{k-j} = c \binom{n}{k-j}$. \square

Proof of Theorem 3 Part 1. Let $\mathcal{F} \subset \binom{[n]}{\geq r}$ be a minimal Turán- $\geq_r(n, k, t)$ -system. We are to show that $|\mathcal{F}| \geq \sum_{i=r}^{j-1} \binom{n}{i}$. Consider $\overline{\mathcal{F}_{< j}} = \cup_{i=r}^{j-1} \binom{[n]}{i} \setminus \mathcal{F}$. For every k -set S of $[n]$,

$$\sum_{i=r}^{j-1} \binom{k}{i} = t \leq |\mathcal{F}(S)| = |\mathcal{F}_{< j}(S)| + |\mathcal{F}_{\geq j}(S)| = \sum_{i=r}^{j-1} \binom{k}{i} - |\overline{\mathcal{F}_{< j}}(S)| + |\mathcal{F}_{\geq j}(S)|.$$

Therefore $|\overline{\mathcal{F}_{< j}}(S)| \leq |\mathcal{F}_{\geq j}(S)|$. Consequently (using $\binom{n-x}{k-x}$ is decreasing in x for $0 \leq x \leq k$), $|\overline{\mathcal{F}_{< j}}| \binom{n-j}{k-j} < \sum_{S \in \binom{[n]}{k}} |\overline{\mathcal{F}_{< j}}(S)| \leq \sum_{S \in \binom{[n]}{k}} |\mathcal{F}_{\geq j}(S)| \leq |\mathcal{F}_{\geq j}| \binom{n-j}{k-j}$. Thus $|\overline{\mathcal{F}_{< j}}| \leq |\mathcal{F}_{\geq j}|$, and therefore $|\mathcal{F}| = |\mathcal{F}_{< j}| + |\mathcal{F}_{\geq j}| \geq |\mathcal{F}_{< j}| + |\overline{\mathcal{F}_{< j}}| = \sum_{i=r}^{j-1} \binom{n}{i}$. \square

The main tool to prove the second part of Theorem 3 is the following well-known fact. For a family \mathcal{G} of r -graphs, the extremal function $\text{ex}(n, \mathcal{G})$ is the maximum number of edges in an r -graph on n vertices that contains no copy of any member of \mathcal{G} .

Theorem 5 (Erdős-Simonovits [2]). *For every $\varepsilon > 0$ and every family of r -graphs \mathcal{G} , each of whose members has k vertices, there exists $\delta > 0$, such that every r -graph on n vertices with at least $\text{ex}(n, \mathcal{G}) + \varepsilon \binom{n}{r}$ edges contains at least $\delta \binom{n}{k}$ copies of members of \mathcal{G} .*

Proof of Theorem 3 Part 2. It suffices to show that for every $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon, k, t) > 0$, such that for all $n \geq n_0$, $T_{\geq r}(n, k, t) \geq (1 - \varepsilon) \left(\sum_{i=r}^{j-1} \binom{n}{i} + T_j(n, k, t_0) \right)$. In fact, this follows from the following claims (taking $n_0 = \max\{n_1, n_2\}$):

- (a) $T_{\geq r}(n, k, t) \geq T_{\geq j}(n, k, t_0)$,
- (b) $T_{\geq j}(n, k, t_0) > (1 - \varepsilon/2)T_j(n, k, t_0)$ for $n > n_1$,
- (c) $T_j(n, k, t_0) \geq T_j(n, k, 1) \geq \binom{n}{j} / \binom{k}{j} > \frac{2(1-\varepsilon)}{\varepsilon} \sum_{i=r}^{j-1} \binom{n}{i}$, for $n > n_2$.

Since (a) and (c) are easy to see, we only prove (b). Suppose that \mathcal{F} is a Turán- $\geq j(n, k, t_0)$ -system. Let \mathcal{G} be the family of all j -graphs on k vertices with more than $\binom{k}{j} - t_0$ edges. Let δ be the output of Theorem 5 for inputs $\varepsilon/[2\binom{k}{j}]$ and \mathcal{G} , and choose n_1 so that $\delta \binom{n}{k} > n^{k-1}$ for all $n > n_1$ (note that $n_1 = n_1(\varepsilon, k, t)$). We will show that $|\mathcal{F}_j| > (1 - \varepsilon/2)T_j(n, k, t_0)$ for $n > n_1$. Suppose, for contradiction, that $|\mathcal{F}_j| \leq (1 - \varepsilon/2)T_j(n, k, t_0)$. Since $\text{ex}(n, \mathcal{G}) = \binom{n}{j} - T_j(n, k, t_0)$ and $T_j(n, k, t_0) \geq \binom{n}{j} / \binom{k}{j}$,

$$|\overline{\mathcal{F}}_j| \geq \binom{n}{j} - \left(1 - \frac{\varepsilon}{2}\right) T_j(n, k, t_0) = \text{ex}(n, \mathcal{G}) + \frac{\varepsilon}{2} T_j(n, k, t_0) \geq \text{ex}(n, \mathcal{G}) + \frac{\varepsilon}{2 \binom{k}{j}} \binom{n}{j}.$$

By Theorem 5 applied with input $\varepsilon/[2\binom{k}{j}]$, the j -graph with vertex set $[n]$ and edge set $\overline{\mathcal{F}}_j$ contains at least $\delta \binom{n}{k}$ copies of (not necessarily the same) members of \mathcal{G} . In other words, there are at least $\delta \binom{n}{k}$ k -sets of $[n]$ that contain fewer than t_0 members of \mathcal{F}_j .

Now consider the family of k -sets of $[n]$ which contains at least one member of \mathcal{F}_i for some $i > j$. Denote this by \mathcal{K}_i and let $\mathcal{K} = \cup_{j < i \leq k} \mathcal{K}_i$. Since $|\mathcal{K}_i| \leq |\mathcal{F}_i| \binom{n-i}{k-i}$ and $|\mathcal{F}| \leq \binom{n}{j}$,

$$|\mathcal{K}| = \sum_{j < i \leq k} |\mathcal{K}_i| \leq \sum_{j < i \leq k} |\mathcal{F}_i| \binom{n-i}{k-i} \leq \binom{n-j-1}{k-j-1} |\mathcal{F}| \leq \binom{n-j-1}{k-j-1} \binom{n}{j} < n^{k-1}.$$

Since $\delta \binom{n}{k} > n^{k-1} > |\mathcal{K}|$ for $n > n_1$, at least one k -set S of $[n]$ contains fewer than t_0 members of \mathcal{F}_j and no member of \mathcal{F}_i for $i > j$. Consequently S contains fewer than t_0 members of \mathcal{F} . This contradicts the assumption that \mathcal{F} is a Turán- $\geq j(n, k, t_0)$ -system. \square

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