

Forbidding complete hypergraphs as traces

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Abstract

Let $2 \leq q \leq \min\{p, t - 1\}$ be fixed and $n \rightarrow \infty$. Suppose that \mathcal{F} is a p -uniform hypergraph on n vertices that contains no complete q -uniform hypergraph on t vertices as a trace. We determine the asymptotic maximum size of \mathcal{F} in many cases. For example, when $q = 2$ and $p \in \{t, t + 1\}$, the maximum is $(\frac{n}{t-1})^{t-1} + o(n^{t-1})$, and when $p = t = 3$, it is $\lfloor \frac{(n-1)^2}{4} \rfloor$ for all $n \geq 3$. Our proofs use the Kruskal-Katona theorem, an extension of the sunflower lemma due to Füredi, and recent results on hypergraph Turán numbers.

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1. Introduction

Let $[n] = \{1, 2, \dots, n\}$. Given a set X , 2^X denotes the family of all subsets of X , and $\binom{X}{q} = \{A \subseteq X : |A| = q\}$. A *hypergraph* \mathcal{H} on X is a family of subsets of X ; these subsets are called *edges* of \mathcal{H} and X is the *vertex* set of \mathcal{H} . If all edges of \mathcal{H} have size p , then \mathcal{H} is a p -uniform hypergraph (p -graph for short).

Let G be a hypergraph on X and $S \subseteq X$. We define the *trace* of G on S as

$$G|_S := \{E \cap S : E \in G\}.$$

Note that we omit multiplicity when defining $G|_S$.

If there exists a set S such that $G|_S$ contains a copy of F as a subhypergraph, we say that G contains F as a trace, or F is a trace of G . In this case we write $G \rightarrow F$, otherwise $G \not\rightarrow F$. Let $L^p(n, F)$ ($L(n, F)$) denote the maximum number of edges in a p -uniform (not necessarily uniform) hypergraph on $[n]$ not containing F as a trace. Extremal problems on traces started from determining $L(n, 2^{[t]})$. Sauer [15], Perles-Shelah [16], and Vapnik-Chervonenkis [18] independently found that $L(n, 2^{[t]}) = \binom{n}{0} + \dots + \binom{n}{t-1}$. For the uniform case, Frankl and Pach [6] showed that $L^p(n, 2^{[t]}) \leq \binom{n}{t-1}$ for $t \leq p \leq n$. Many intersecting problems and applications on traces can be found in the survey of Füredi and Pach [9].

In this paper we consider the problem of forbidding a level of the lattice $2^{[t]}$ as a trace. More precisely, given integers p, t, n with $\max\{p, t\} \leq n$, we study the value of $L^p(n, \binom{[t]}{q})$ for $1 \leq q \leq t-1$ (the $q=0$ and $q=t$ cases are trivial). Frankl and Pach [6] studied the $q=1$ case and obtained that $\text{ex}(p+t-1, \binom{[t]}{t-1}) \leq L^p(n, \binom{[t]}{1}) \leq \binom{p+t-1}{t-1}$, where ex is the classical Turán number. Balogh, Keevash and Sudakov [1] investigated the trace problem of forbidding more than one non-trivial level of $2^{[t]}$.

Trivially $L^p(n, \binom{[t]}{q}) = \binom{n}{p}$ when $p < q$. Therefore throughout the paper we assume that

$$2 \leq q \leq t-1 \quad \text{and} \quad q \leq p, \tag{1}$$

and whenever we use asymptotic notation, we assume that only $n \rightarrow \infty$. Note that the $p=q$ case is exactly the Turán problem. The reason why we only consider *uniform* trace numbers is that Füredi and Quinn [10] showed that $L(n, \binom{[t]}{q}) = L(n, 2^{[t]})$ for every $0 \leq q \leq t$, in other words, forbidding a level of the lattice $2^{[t]}$ is equivalent to forbidding the whole lattice in the non-uniform case. Following graph theory language, the forbidden configuration $\binom{[t]}{q}$ is a complete q -graph on t vertices, so we denote it by K_t^q , and write $K_t = K_t^2$.

Our first result, which is little more than an observation, determines the order of magnitude of $L^p(n, K_t^q)$.

Proposition 1.1. $L^p(n, K_t^q) = \Theta(n^{\min\{p, t-1\}})$.

A trace problem for uniform hypergraphs is in fact a Turán problem. Given a family \mathcal{F} of r -graphs, the *Turán number* $\text{ex}(n, \mathcal{F})$ of \mathcal{F} is the maximum number of edges in an r -graph on n vertices containing no $F \in \mathcal{F}$ (see *e.g.*, Füredi [8] for a survey). When $\mathcal{F} = \{F\}$, we write $\text{ex}(n, F)$ instead of $\text{ex}(n, \{F\})$. If we denote K_3 by $\{12, 23, 31\}$, then $L^3(n, K_3) = \text{ex}(n, \{F_1, F_2, F_3\})$, where $F_1 = \{124, 234, 134\}$, $F_2 = \{124, 234, 135\}$, and $F_3 = \{124, 235, 136\}$. In general, for any q -graph F and $q \leq p$, we have $L^p(n, F) = \text{ex}(n, \mathcal{H}^p(F))$, where $\mathcal{H}^p(F)$ is the family of all p -graphs H with $|F|$ edges such that $H \rightarrow F$.

Definition 1.2. Let $H_{q,t}^p$ be the member of $\mathcal{H}^p(K_t^q)$ with the maximum number of vertices. In other words, $H_{q,t}^p$ is the p -graph obtained from K_t^q by enlarging each of its $\binom{t}{q}$ edges with a (different) set of $p - q$ new vertices. Trivially $H_{p,t}^p = K_t^p$.

Since forbidding a family of hypergraphs (as a subgraph) is not easier than forbidding any member of the family,

$$L^p(n, K_t^q) = \text{ex}(n, \mathcal{H}^p(K_t^q)) \leq \text{ex}(n, H_{q,t}^p). \quad (2)$$

Our second result, which is also not hard to prove, shows that the inequality in (2) is asymptotically an equality when $p < t$.

Proposition 1.3. Let $p < t$. Then $L^p(n, K_t^q) = \text{ex}(n, H_{q,t}^p) + o(n^p)$.

Our main result reduces $L^p(n, K_t^q)$ when $p \geq t$ to Turán numbers in many cases.

Theorem 1.4. Fix $2 \leq q < t \leq p$. Suppose that $q \in \{t-2, t-1\}$ or $p \in \{t, t+1\}$. Then

$$L^p(n, K_t^q) = L^{t-1}(n, K_t^q) + o(n^{t-1}) = \text{ex}(n, H_{q,t}^{t-1}) + o(n^{t-1}). \quad (3)$$

This suggests that determining $L^p(n, K_t^q)$ could be as difficult as a hypergraph Turán problem. For example, (3) implies that $L^4(n, K_4^3) = \text{ex}(n, K_4^3) + o(n^3)$, and determining $\text{ex}(n, K_4^3)$ is a well-known open problem of Turán [17]. Together with Mantel's Theorem on $\text{ex}(n, K_3)$ [12], Theorem 1.4 gives

$$L^p(n, K_3) = \text{ex}(n, K_3) + o(n^2) = \left(\frac{n}{2}\right)^2 + o(n^2). \quad (4)$$

Determining $\text{ex}(n, H_{q,t}^p)$ in general seems hopeless. However, the $q = 2$ case was recently solved by the first author [13] and Pikhurko [14]. Given $2 \leq p \leq \ell$, a p -graph is ℓ -partite if its vertices can be partitioned into ℓ classes, such that every edge has at most one vertex from each class. An ℓ -partite p -graph is called *complete* if it contains all allowable edges. We denote by $T_\ell^p(n)$ the complete ℓ -partite p -graph (a *generalized Turán graph*) on n vertices with no two class sizes differ more than one. Let $p < t$. Clearly $T_{t-1}^p(n)$ contains no $H_{2,t}^p$ as a subgraph and

$$|T_\ell^p(n)| = \sum_{S \in \binom{[\ell]}{p}} \prod_{i \in S} \left\lfloor \frac{n+i-1}{\ell} \right\rfloor = \binom{\ell}{p} \left(\frac{n}{\ell}\right)^p + o(n^p).$$

The first author [13] showed that $\text{ex}(n, H_{2,t}^p) = |T_{t-1}^p(n)| + o(n^p)$ as $n \rightarrow \infty$. Pikhurko [14] improved this to $\text{ex}(n, H_{2,t}^p) = |T_{t-1}^p(n)|$ for sufficiently large n . Applying (2), we thus have $L^p(n, K_t) \leq |T_{t-1}^p(n)|$ for sufficiently large n . On the other hand, it is easy to see that $T_{t-1}^p(n)$ contains no K_t^q for any $q \geq 2$ as a trace. In fact, every t -vertex set S of $T_{t-1}^p(n)$ must contain two vertices a, b from the same vertex class, but no edge of $T_{t-1}^p(n)$ contains both a and b . Thus for $q \geq 2$, every q -subset of S containing a and b is absent from $T_{t-1}^p(n)|_S$. Consequently $T_{t-1}^p(n) \not\rightarrow K_t^q$, in particular, $L^p(n, K_t) \geq |T_{t-1}^p(n)|$. Putting the upper and lower bounds together, for $2 \leq p < t$ and sufficiently large n ,

$$L^p(n, K_t) = |T_{t-1}^p(n)| = \text{ex}(n, H_{2,t}^p). \quad (5)$$

By combining (5) with Theorem 1.4, we obtain the following result.

Corollary 1.5. *Suppose that $t = 4$ or $p \in \{t, t + 1\}$. Then*

$$L^p(n, K_t) = |T_{t-1}^{t-1}(n)| + o(n^{t-1}) = \left(\frac{n}{t-1}\right)^{t-1} + o(n^{t-1}). \quad (6)$$

We conjecture the values of $L^p(n, K_t^q)$ as follows.

Conjecture 1.6. *Fix p, q, t, n with $2 \leq q < \min\{t, p\}$. For $n > n_0$,*

$$L^p(n, K_t^q) = \begin{cases} \text{ex}(n, H_{q,t}^p) & \text{if } p < t, \\ L^{t-1}(n-p+t-1, K_t^q) = \text{ex}(n-p+t-1, H_{q,t}^{t-1}) & \text{if } p \geq t. \end{cases}$$

The equation (5) confirms the conjecture for the case of $q = 2$, $p < t$, and sufficiently large n . As further evidence of Conjecture 1.6, we prove its smallest non-trivial case: $(p, q, t) = (3, 2, 3)$. Note that this sharpens the $p = 3$ case of (4).

Theorem 1.7. *Let $n \geq 3$. Then*

$$L^3(n, K_3) = \text{ex}(n-1, K_3) = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor.$$

2. Preliminary Results

In this section we prove Proposition 1.1, Proposition 1.3 and the supersaturation property for trace problems.

We first observe that $L^p(n, K_t^q)$ is close to a monotone function of n .

Proposition 2.1. $L^p(n, K_t^q) \geq L^{p-i}(n-i, K_t^q)$ for $1 \leq i \leq p-q$.

Proof. Suppose that $G \subseteq \binom{[n-i]}{p-i}$ satisfies $G \not\rightarrow K_t^q$. We extend G to a p -graph G' by adding a set C of i new vertices and replacing each $E \in G$ by $E \cup C$. We claim that $G' \not\rightarrow K_t^q$. Consider a t -set S of vertices. If S contains a vertex $x \in C$, then all edges of G' contain x , and consequently all q -subsets of $E \setminus \{x\}$ are absent from $G'|_E$. Otherwise $E \cap C = \emptyset$, and we have $G'|_E = G|_E \not\rightarrow K_t^q$. \square

Proof of Proposition 1.1. We need to show that $L^p(n, K_t^q) = \Theta(n^{\min\{p, t-1\}})$. When $p \geq t$, Frankl and Pach [6] showed that $L^p(n, K_t^q) \leq \binom{n}{t-1}$. When $p < t$, trivially $L^p(n, K_t^q) \leq \binom{n}{p}$. We now consider lower bounds. When $p \leq t-1$, since $T_{t-1}^p(n) \not\rightarrow K_t^q$ (for $q \geq 2$), we have $L^p(n, K_t^q) \geq |T_{t-1}^p(n)| = \Omega(n^p)$. Now let $p \geq t$. Since $T_{t-1}^{t-1}(n-p+t-1) \not\rightarrow K_t^q$ and $|T_{t-1}^{t-1}(n-p+t-1)| = \Omega(n^{t-1})$, we have $L^{t-1}(n-p+t-1, K_t^q) = \Omega(n^{t-1})$. Proposition 2.1 thus implies that $L^p(n, K_t^q) \geq L^{t-1}(n-p+t-1, K_t^q) = \Omega(n^{t-1})$. \square

Definition 2.2. Let F be a p -graph ($p \geq 2$) on $[\ell]$ and $\vec{m} = \langle m_1, \dots, m_\ell \rangle$ be a vector of positive integers. The blow-up $F(\vec{m})$ of F is obtained by replacing each vertex i by a vertex class V_i of size m_i , and each edge $\{i_1, \dots, i_p\}$ by the family of all p -sets $\{w_1, \dots, w_p\}$, where $w_j \in V_{i_j}$. We simply write $F(m)$ if all $m_i = m$.

A phenomenon discovered by Brown, Erdős and Simonovits [5], usually called *supersaturation*, implies that $\text{ex}(n, F(\vec{m})) = \text{ex}(n, F) + o(n^r)$ for every r -graph F and its blow-up $F(\vec{m})$. To prove Proposition 1.3, we need a lemma from [13], which is a simple consequence of supersaturation.

Lemma 2.3 (Lemma 4 in [13]). *Let m, p be positive integers with $p \geq 2$, and let \mathcal{F} be a finite family of p -graphs. If H is a p -graph satisfying $H \subseteq F(m)$ for all $F \in \mathcal{F}$, then $\text{ex}(n, H) \leq \text{ex}(n, \mathcal{F}) + o(n^p)$. \square*

Proof of Proposition 1.3. Here $p < t$ and we must show that $L^p(n, K_t^q) = \text{ex}(n, H_{q,t}^p) + o(n^p)$. Because of (2), we only need to show that $L^p(n, K_t^q) \geq \text{ex}(n, H_{q,t}^p) + o(n^p)$ when $n \rightarrow \infty$. For each $F \in \mathcal{H}^p(K_t^q)$, it is easy to see that $H_{q,t}^p \subseteq F(\binom{[t]}{q})$. Lemma 2.3 implies that

$$\text{ex}(n, H_{q,t}^p) \leq \text{ex}(n, \mathcal{H}^p(K_t^q)) + o(n^p) = L^p(n, K_t^q) + o(n^p). \quad \square \quad (7)$$

Next we prove the supersaturation phenomenon for trace problems.

Lemma 2.4. $L^p(n, K_t^q(m)) \leq L^p(n, K_t^q) + o(n^p)$. *In particular, $L^p(n, K_t^q(m)) = (1 + o(1))L^p(n, K_t^q)$ for $p < t$.*

Proof. The second assertion follows from the first by realizing that $L^p(n, K_t^q) = \Theta(n^p)$ for $p < t$ from Proposition 1.1. To prove the first claim, recall that $\mathcal{H}^p(K_t^q(m))$ is the family of p -graphs whose $|K_t^q(m)|$ edges contain $K_t^q(m)$ as a trace, and $H_{q,t}^p$ is obtained from K_t^q by enlarging each of its $\binom{[t]}{q}$ edges with a different set of $p - q$ new vertices. Let $\tilde{H} = H_{q,t}^p(\vec{m})$, where $m_i = m$ for all the vertices v_i in the original K_t^q , and $m_i = 1$ for the other vertices. It is easy to see that \tilde{H} is a member of $\mathcal{H}^p(K_t^q(m))$. We thus have

$$\begin{aligned} L^p(n, K_t^q(m)) &= \text{ex}(n, \mathcal{H}^p(K_t^q(m))) \\ &\leq \text{ex}(n, \tilde{H}) \\ &\leq \text{ex}(n, H_{q,t}^p) + o(n^p) \\ &\leq L^p(n, K_t^q) + o(n^p), \end{aligned}$$

where the first inequality holds because $\tilde{H} \in \mathcal{H}^p(K_t^q(m))$, the second inequality holds because of supersaturation for the Turán problems, and the last one holds because of (7). \square

3. Proof of Theorem 1.4

Throughout this section we will assume that $p \geq t$. Our goal is to prove that if $q \in \{t - 2, t - 1\}$ or $p \in \{t, t + 1\}$, then

$$L^p(n, K_t^q) = L^{t-1}(n, K_t^q) + o(n^{t-1}).$$

In fact, the second equality of (3) in Theorem 1.4, $L^{t-1}(n, K_t^q) = \text{ex}(n, H_{q,t}^{t-1}) + o(n^{t-1})$, follows from Proposition 1.3 (note that the second condition in (1) still holds because $t-1 \geq q$). Furthermore, we claim that

$$L^p(n, K_t^q) \geq L^{t-1}(n, K_t^q) + o(n^{t-1}). \quad (8)$$

To see this, first observe that Proposition 2.1 implies that $L^p(n, K_t^q) \geq L^{t-1}(n-p+t-1, K_t^q)$. Proposition 1.3 further gives that $L^p(n, K_t^q) \geq \text{ex}(n-p+t-1, H_{q,t}^{t-1}) + o(n^{t-1})$. Now we recall a fact on the Turán number, which immediately follows from the existence of $\lim_{n \rightarrow \infty} \text{ex}(n, \mathcal{F}) / \binom{n}{p}$. Given a family \mathcal{F} of r -graphs and an integer $c > 0$,

$$\text{ex}(n, \mathcal{F}) - \text{ex}(n-c, \mathcal{F}) = o(n^r). \quad (9)$$

Therefore $L^p(n, K_t^q) \geq \text{ex}(n, H_{q,t}^{t-1}) + o(n^{t-1})$ and (8) follows after applying Proposition 1.3 again.

Therefore the main task is to verify

$$L^p(n, K_t^q) \leq L^{t-1}(n, K_t^q) + o(n^{t-1}). \quad (10)$$

for $q \in \{t-2, t-1\}$ or $p \in \{t, t+1\}$. The $q = t-1$ case (Section 3.1) is the easiest: its main idea is to find a one-to-one function from a p -graph G with $G \not\rightarrow K_t^q$ to a $(t-1)$ -graph G' such that $G' \not\rightarrow K_t^{t-1}$. The remaining cases are harder: we present two lemmas in Section 3.2, and complete the proofs in Section 3.3. The main tools include the Erdős-Ko-Rado theorem, the Kruskal-Katona theorem and a lemma on sunflowers due to Füredi.

3.1. $q = t - 1$.

Let G be a hypergraph and S be a subset of its vertex set. The *degree* of S in G , $\deg_G(S)$, or $\deg(S)$ if the underlying hypergraph is clear from the context, is the number of edges in G containing S (frequently called codegree when $|S| \geq 2$). Given a p -graph G , if every edge $E \in G$ contains at least one p' -subset E' with $\deg_G(E') = 1$, then $\phi(E) = E'$ defines a one-to-one function from G to $G' = \{E' : E \in G\}$ (if more than one p' -subsets are of degree 1, then arbitrarily pick one of them to be $\phi(E)$).

Proposition 3.1. *Let G be a p -graph such that $G \not\rightarrow K_t^q$. If there exists a function ϕ mapping every edge $E \in G$ to a q -set $E' \subset E$ such that $\deg_G(E') = 1$, then $\phi(G) = \{\phi(E) : E \in G\}$ contains no K_t^q as a subgraph.*

Proof. Suppose instead, that G' contains a subgraph G'_1 on a t -set T such that $G_1 \cong K_t^q$. Clearly ϕ is one-to-one. Let ϕ^{-1} be the inverse function. We claim that each edge $E \in G$ with $\phi(E) \in G'_1$ satisfies that $E \cap T = \phi(E)$ and therefore $G|_T \supseteq K_t^q$, contradicting the assumption that $G \not\rightarrow K_t^q$. In fact, if $E \cap T \supsetneq \phi(E)$, then $E \cap T$ contains another q -set $Q \in G'_1$. Clearly $E \neq \phi^{-1}(Q)$ because ϕ is a function. The fact that both E and $\phi^{-1}(Q)$ contain Q implies that $\deg_G(Q) \geq 2$, a contradiction. \square

The following lemma is the key observation for proving the $q = t - 1$ case of (10).

Lemma 3.2. *Let $2 \leq t \leq p$. Suppose that S is a p -set and H is a family of proper subsets of S . If every $(t - 1)$ -subset of S is contained in some member of H , then $H \rightarrow K_t^{t-1}$.*

Proof. We do induction on p for fixed $t \geq 2$. The base case $p = t$ is trivial, since every $(t - 1)$ -subset of S is a member of H , or $\binom{S}{p-1} \subseteq H$. For the induction step, let $p > t$ and consider two cases. If $\binom{S}{p-1} \subseteq H$, then for a fixed t -set $T \subset S$, we have $H|_T \supseteq \binom{T}{t-1}$ because each $(t - 1)$ -subset T' of T is contained in $T' \cup (S \setminus T)$, which is a member of H . Otherwise $\binom{S}{p-1} \not\subseteq H$, and there exists an $(p - 1)$ -set $S' \notin H$. It is easy to see that $H|_{S'}$ satisfies the assumption of the lemma with $p - 1$ instead of p . We then apply the induction hypothesis to S' and $H|_{S'}$ obtaining that $H|_{S'} \rightarrow K_t^{t-1}$ and consequently $H \rightarrow K_t^{t-1}$. \square

Proof of (10) for $q = t - 1$. Let G be an n -vertex p -graph not having K_t^{t-1} as a trace. Each edge $E \in G$ must contain a $(t - 1)$ -subset E' with $\deg_G(E') = 1$, otherwise we apply Lemma 3.2 with $S = E$ and $H = G|_E - \{E\}$ to conclude that $G \rightarrow K_t^{t-1}$. We thus define $\phi(E) = E'$ and ϕ is a one-to-one function from G to $\binom{[n]}{t-1}$. By Proposition 3.1, the resulting $(t - 1)$ -graph G' contains no K_t^{t-1} as a subgraph, thus $|G| = |G'| \leq \text{ex}(n, K_t^{t-1}) = L^{t-1}(n, K_t^{t-1})$. \square

3.2. Two Lemmas

Fix $G \subseteq \binom{[n]}{p}$ with $|G| \geq 2$. The following partition of G will be needed in our proofs. Define a function $f : G \rightarrow [p]$ such that for $E \in G$,

$$f(E) = \min\{|D| : D \subseteq E, \deg(D) = 1 \text{ and } \forall S \subset D, \deg(S) \geq 2\}.$$

(Throughout this subsection $\deg = \deg_G$.) Since $\deg(E) = 1$ and $\deg(\emptyset) = |G| \geq 2$, there always exists a subset $D \subset E$ such that $\deg(D) = 1$ but $\deg(S) \geq 2$ for all $S \subset D$. Hence f

is well defined. For $1 \leq i \leq p$, let $G_i = \{E \in G : f(E) = i\}$. Clearly, $G_p + G_{p-1} + \dots + G_1$ is a partition of G .

Furthermore, for $k \leq p$, let $\partial^k G$ denote the shadow of G at level k , namely, $\partial^k G = \{D : |D| = k, D \subseteq E \text{ for some } E \in G\}$. In particular, $\partial G = \partial^{p-1} G$. Let

$$G^i = \{D \in \partial^i G : \deg(D) = 1 \text{ and } \forall S \subset D, \deg(S) \geq 2\}.$$

If we map each $D \in G^i$ to the unique $E \in G$ such that $D \subseteq E$, then we obtain an onto function from G^i to G_i . Hence $|G_i| \leq |G^i|$ for $1 \leq i \leq p$. We are ready to state two lemmas, which are the key ingredients in our proofs.

Lemma 3.3. *Let $t \leq k \leq p$ and $G \subseteq \binom{[n]}{p}$. If $G \not\approx K_t^q$, then $|\partial^t(G^k)| = O(n^{t-2})$.*

Lemma 3.4. *Let $t \leq i \leq k \leq p$ and $G \subseteq \binom{[n]}{p}$. If $G \not\approx K_t^{t-2}$, then $|\partial^i(G^k)| = O(n^{t-2})$.*

In order to prove Lemma 3.3. We need the following lemma on sunflowers, which is an easy corollary of a result of Füredi [7] and the Erdős-Ko-Rado Theorem [4]. A *sunflower* (or Δ -system) with k petals and a core Y is a collection of distinct sets S_1, \dots, S_k such that $S_i \cap S_j = Y$ for all $i \neq j$.

Lemma 3.5. *Given k and r , there exists $C = C(k, r)$ such that every $F \subseteq \binom{[n]}{k}$ with $|F| \geq Cn^{k-i}$ contains an r -petal sunflower with a core of size less than i .*

Proof. Füredi [7] extended the well-known Sunflower Lemma of Erdős and Rado [3] as follows: *given k and r , there exists $c = c(k, r)$ such that every $F \subseteq \binom{[n]}{k}$ contains a subfamily F' such that $|F'| > c|F|$ and for all distinct $E_1, E_2 \in F'$, F' contains an r -petal sunflower with core $E_1 \cap E_2$. (The original statement in [7] is actually stronger.) Let $C = 1/c$. We apply this result to $F \subseteq \binom{[n]}{k}$ with $|F| \geq n^{k-i}/c$. Since $|F'| \geq n^{k-i} > \binom{n-i}{k-i}$, by the Erdős-Ko-Rado Theorem [4], F' contains E_1, E_2 such that $|E_1 \cap E_2| < i$. Then F' contains an r -petal sunflower with core $E_1 \cap E_2$ of size less than i . \square*

Fix $i \in [p]$. We say that a hypergraph $H \subseteq \partial^i G$ satisfies the property (\diamond) if

$$\text{for all } D \in H \text{ and } x \in D, \text{ there exists } E \in G \text{ s.t. } D \setminus \{x\} \subset E, x \notin E.$$

We claim that $\partial^i G^k$ satisfies (\diamond) for all $t \leq i \leq k$. First we show that G^k satisfies (\diamond) . Pick $D \in G^k$ and $x \in D$. Since $D \in G^k$, there exists a unique $E_1 \in G$ such that $D \subseteq E_1$. Since $\deg(D \setminus \{x\}) \geq 2$, there exists $E \in G$, $E \neq E_1$ such that $D \setminus \{x\} \subset E$. In addition, $x \notin E$,

otherwise $D \subseteq E$, contradicting $\deg(D) = 1$. We next observe that if H satisfies (\diamond) , then ∂H also satisfies (\diamond) . In fact, let $S \in \partial H$ and $x \in S$. Suppose that $S \subset D \in H$. Then there exists $E \in G$ such that $D \setminus \{x\} \subset E$, $x \notin E$, in particular, $S \setminus \{x\} \subset E$.

Given a function $\phi : A \rightarrow B$ and $y \in B$, let $\phi^{-1}(y) = \{x \in A : \phi(x) = y\}$.

Proof of Lemma 3.3. Let $H = \partial^t(G^k)$. Since $G \not\rightarrow K_t^q$, each $D \in H$ contains at least one q -element subset Q such that $Q \notin G_D$. We denote such a Q by $\psi(D)$ (arbitrarily pick one if more than one set can be chosen). In order to show that $|H| = O(n^{t-2})$, it suffices to show that for each set $Q \in \binom{[n]}{q}$, we have $|\psi^{-1}(Q)| = O(n^{t-2-q})$. Define a $(t-q)$ -graph $F = \{D - Q : D \in \psi^{-1}(Q)\}$. Suppose to the contrary, that $|\psi^{-1}(Q)| = |F| > Cn^{t-q-2}$ for the constant $C = C(t-q, p-t+3)$ from Lemma 3.5. By Lemma 3.5, F contains a sunflower S_1, \dots, S_{p-t+3} with core Y of size at most 1. For all i , let $D_i = S_i \cup Q \in H$.

Case 1: $Y = \emptyset$. Since $D_1 \in \partial^t(G^k)$, there exists $E \in G$ such that $D_1 \subset E$. At most $|E \setminus D_1| = p-t$ petals have non-empty intersection with $E \setminus D_1$. Since the total number of petals is greater than $p-t+1$, there exists $j \neq 1$ such that $S_j \cap (E \setminus D_1) = \emptyset$, or $D_j \cap E = Q$, a contradiction.

Case 2: $Y = \{x\}$. Since H satisfies (\diamond) , there exists $E \in G$ such that $D_1 \setminus \{x\} \subset E$ and $x \notin E$. At most $|E \setminus (D_1 \setminus \{x\})| = p-t+1$ petals have non-empty intersection with $E \setminus (D_1 \setminus \{x\})$. Since the total number of petals is $p-t+3$, there exists $j, j \neq 1$ such that $S_j \cap (E \setminus (D_1 \setminus \{x\})) = \emptyset$. Since $x \notin E$ but $x \in D_j$, we have $D_j \cap E = Q$, a contradiction. \square

Proof of Lemma 3.4. We do induction on $i \geq t$. The base case $i = t$ holds because of Lemma 3.3. Let $H = \partial^i G^k$. For each $D \in H$, arbitrarily pick one of its t -subsets S . Since $G \not\rightarrow K_t^{t-2}$, S contains a $(t-2)$ -subset Q such that $Q \notin G|_S$. Suppose $S \setminus Q = \{x, y\}$. Let $\psi(D) = D - \{y\}$ and $\phi(D) = (\psi(D), Q, x)$. We claim that $|\psi^{-1}(D - \{y\})| \leq \binom{i-1}{t-1}(t-1)(p-i+2)$. By the pigeonhole principle, it suffices to show that $|\phi^{-1}(D - \{y\}, Q, x)| \leq p-i+2$ (for a fixed $D - \{y\}$, there are $\binom{i-1}{t-1}(t-1)$ ways of choosing a $(t-2)$ -set Q and an element $x \notin Q$). Suppose instead, that there exist $D_1, \dots, D_{p-i+3} \in H$ forming a sunflower with core $D - \{y\}$ and petals $\{y_j\}, 1 \leq j \leq p-i+3$ such that $Q \notin G|_{S_j}$ for $S_j = Q \cup \{x, y_j\}$. Since H satisfies (\diamond) , there exists $E \in G$ such that $D_1 \setminus \{x\} \subset E$ and $x \notin E$. At most $|E \setminus (D_1 \setminus \{x\})| = p-i+1$ petals have non-empty intersection with $E \setminus (D_1 \setminus \{x\})$. Since the total number of petals is $p-t+3$, there exists $j \neq 1$ such that $y_j \notin E$. Since $x \notin E$ but $x \in D_j$, we have $S_j \cap E = Q$, a contradiction.

We thus have $|H| \leq C|\psi(H)|$, where $C = \binom{i-1}{t-1}(t-1)(p-i+2)$. Since $\psi(H) \subseteq \partial^{i-1}G^k$, the

induction hypothesis gives $|\psi(H)| \leq |\partial^{i-1}G^k| = O(n^{t-2})$. Consequently $|H| = O(n^{t-2})$. \square

3.3. Proofs for $p \in \{t, t+1\}$ and $q = t-2$

We need a proposition, which can be considered as an extension of Proposition 3.1.

Proposition 3.6. *Let $q \leq p' \leq p$, and $m = \binom{t}{q}(p-q) + 1$. Suppose that G is a p -graph on $[n]$ and ϕ is a function from G to $\binom{[n]}{p'}$ such that $\phi(E) \subseteq E$ for each $E \in G$. If $G \not\rightarrow K_t^q$, then $\phi(G) \not\rightarrow K_t^q(m)$.*

Proof. Suppose instead, that $\phi(G) \rightarrow K_t^q(m)$. Then there are disjoint vertex sets X_1, X_2, \dots, X_t of size m such that the following holds. Let \mathcal{Q} be the family of q -sets having non-empty intersection with exactly q of X_1, X_2, \dots, X_t . For each $Q \in \mathcal{Q}$, there exists $E \in G$ such that $Q \subseteq \phi(E) \subseteq E$. Denote such E by E_Q . We say that a set $Q \in \mathcal{Q}$ is *bad* if there exists j such that $Q \cap X_j = \emptyset$ and $(E_Q \setminus Q) \cap X_j \neq \emptyset$. Given a bad $Q \in \mathcal{Q}$, a t -tuple x_1, \dots, x_t with $x_i \in X_i$ is called *bad because of Q* if $\{x_1, \dots, x_t\}$ contains Q and at least one vertex from $E_Q \setminus Q$. A t -tuple from $X_1 \times \dots \times X_t$ is called *bad* if it is bad because of some Q . For fixed bad $Q \in \mathcal{Q}$, the number of bad t -tuples because of Q is at most $(p-q)m^{t-q-1}$ (first select a vertex from $E_Q \setminus Q$ and then decide the remaining $t-q-1$ coordinates). The total number of bad t -tuples is thus at most $\binom{t}{q}m^q(p-q)m^{t-q-1}$. When $m > \binom{t}{q}(p-q)$, we have $\binom{t}{q}m^q(p-q)m^{t-q-1} < m^t$, or the number of bad t -tuples is less than the total number of t -tuples in $X_1 \times \dots \times X_t$. Hence there always exists a good t -tuple T and consequently $G|_T \supseteq K_t^q$, a contradiction. \square

Proof of (10) for $p = t$. Given $G \subseteq \binom{[n]}{t}$ such that $G \not\rightarrow K_t^q$, we partition G into $G_t + \dots + G_1$ as in the beginning of Section 3.2. By Lemma 3.3, $|G^t| = O(n^{t-2})$ and consequently $|G_t| \leq |G^t| = O(n^{t-2})$. Trivially $|G_i| \leq |G^i| = O(n^{t-2})$ for $i \leq t-2$. It remains to show that $|G_{t-1}| \leq L^{t-1}(n, K_t^q) + o(n^{t-1})$. In fact, for each $E \in G_{t-1}$, we define $\phi(E) = D$ where D is one of the $(t-1)$ -subsets of E satisfying $\deg(D) = 1$. Proposition 3.6 implies that $\phi(G) \not\rightarrow K_t^q(m)$ for $m = \binom{t}{q}(p-q) + 1$. So

$$|G| = |\phi(G)| \leq L^{t-1}(n, K_t^q(m)) \leq L^{t-1}(n, K_t^q) + o(n^{t-1}),$$

where the last inequality follows from Lemma 2.4.

Proof of (10) for $p = t+1$. We need Lovász's version [11] of the Kruskal-Katona Theorem: *let H be a $(t+1)$ -graph with $|H| = \binom{x}{t+1}$ for some real number x . Then $\partial H \geq \binom{x}{t}$.* This

implies that if $|\partial H| = O(n^k)$, then $|H| = O(n^{\frac{k(t+1)}{t}})$. To see this, suppose that $|\partial H| \leq Cn^k$ for some $C > 0$. Since $\left(\frac{x}{t}\right)^t \leq \binom{x}{t} \leq |\partial H| \leq Cn^k$, we have $\frac{x}{t} \leq C^{\frac{1}{t}} n^{\frac{k}{t}}$ and

$$|H| = \binom{x}{t+1} = \binom{x}{t} \frac{x-t}{t+1} \leq Cn^k C^{\frac{1}{t}} n^{\frac{k}{t}} = O(n^{\frac{k(t+1)}{t}}).$$

Now given $G \subseteq \binom{[n]}{t+1}$ such that $G \not\subseteq K_t^q$, we partition G into $G_{t+1} + G_t + \dots + G_1$. The proof of the $p = t$ case shows $\sum_{i=1}^t |G_i| \leq L^{t-1}(n, K_t^q) + o(n^{t-1})$. It suffices to show that $|G_{t+1}| = o(n^{t-1})$, or $|G^{t+1}| = o(n^{t-1})$. Lemma 3.3 guarantees that $\partial^t(G^{t+1}) = O(n^{t-2})$ and consequently, by the result of Lovász, $|G^{t+1}| = O(n^{\frac{(t-2)(t+1)}{t}}) = o(n^{t-1})$. \square

Proof of (10) for $q = t - 2$. Given $G \subseteq \binom{[n]}{p}$ such that $G \not\subseteq K_t^{t-2}$, we partition G into $G_p + \dots + G_t + G_{t-1} + \dots + G_1$. The proof of the $p = t$ case shows $\sum_{i=1}^t |G_i| \leq L^{t-1}(n, K_t^q) + o(n^{t-1})$. For $t < k \leq p$, we apply Lemma 3.4 with $i = k$ and obtain that $|G_k| \leq |G^k| \leq \partial^k(G^k) = O(n^{t-2})$, thus completing the proof. \square

4. An Exact Result

In order to prove Theorem 1.7, we need the following lemma, which can be proved by following the original proof of Mantel's Theorem [12]. We use $+$ instead of \cup for a disjoint union. In a graph G , given a vertex set A and a vertex x , $N(x, A)$ denotes the neighborhood of x in A , and $d(x, A) = |N(x, A)|$, in particular $d(x) = d(x, V(G))$. For disjoint vertex sets X and Y , we denote by $e(X, Y)$ the number of edges between X and Y . For simplicity we write ab instead of $\{a, b\}$.

Lemma 4.1. *Let $G = (V, E)$ be a triangle-free graph such that*

$$\text{for every } ab \in E, \text{ there exists } c \in V, \text{ such that } ac \notin E \text{ and } bc \notin E. \quad (\star)$$

Then $|E| \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1$ with equality only when G has the following structure: $V(G) = A + B + \{z\}$, there exist $a \in A$ and a non-empty set $B_z \subseteq B$ such that $E(G) = A \times B - \{ab : b \in B_z\} + \{zb : b \in B_z\} + \{az\}$.

Proof. Let xy be an edge. Since G is triangle-free, we have $N(x) \cap N(y) = \emptyset$. With (\star) , we further derive that $d(x) + d(y) \leq n - 1$.

If $d(x) + d(y) \leq n - 2$ for every edge xy in G , then following Mantel's proof of his theorem, we have

$$\frac{4|E|^2}{n} = \frac{(\sum_{x \in V} d(x))^2}{n} \leq \sum_{x \in V} (d(x))^2 = \sum_{xy \in E} (d(x) + d(y)) \leq (n - 2)|E|,$$

$$|E| \leq \frac{n(n - 2)}{4} < \frac{(n - 1)^2}{4} < \left\lfloor \frac{(n - 1)^2}{4} \right\rfloor + 1.$$

Otherwise assume that $d(x) + d(y) = n - 1$ for some $e = \{x, y\}$. Let $A = N(y)$ and $B = N(x)$. We know that $A \cap B = \emptyset$ and $A \cup B = V - \{z\}$ for some vertex z . Let $d_1 = d(z, A)$ and $d_2 = d(z, B)$.

Case 1: $d_1 = 0$, or $d_2 = 0$.

Say, $d_1 = 0$. For each $b \in N(z, B)$, there exists $a \in A$ such that $ab \notin E$, since otherwise edge xb does not satisfy (\star) . This implies that

$$|E| = e(A, B) + d(z, B) \leq |A||B| \leq \left\lfloor \frac{(n - 1)^2}{4} \right\rfloor.$$

Case 2: $d_1, d_2 > 0$.

In this case $d_1 d_2 - d_1 - d_2 + 1 = (d_1 - 1)(d_2 - 1) \geq 0$ with equality if and only if at least one of d_1, d_2 is 1. Since G is triangle-free, there is no edge between $N(z, A)$ and $N(z, B)$. Thus $e(A, B) \leq |A||B| - d_1 d_2$ and

$$|E| = e(A, B) + d(z, A) + d(z, B) \leq |A||B| - d_1 d_2 + d_1 + d_2 \leq \left\lfloor \frac{(n - 1)^2}{4} \right\rfloor + 1,$$

where equality holds only when G has the desired structure. \square

Proof of Theorem 1.7. To show that $L^3(n, K_3) \geq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$, we enlarge each edge of $K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$ with the same new vertex.

To prove the upper bound, we consider a 3-graph H on $[n]$ such that $H \not\rightarrow K_3$. The proof of the $q = t - 1$ case of Theorem 1.4 implies that each triple $T \in H$ contains a pair $\phi(T)$ with $\deg_H(\phi(T)) = 1$. We thus obtain a graph G on $[n]$ with edge set $E = \{\phi(T) : T \in H\}$. Clearly $|E| = |H|$, and G satisfies (\star) because

$$\text{if } \phi(\{a, b, c\}) = ab, \text{ then } ac \notin E \text{ and } bc \notin E. \quad (11)$$

Next we claim that $G \neq G^*$, where G^* is a graph causing the equality in Lemma 4.1. Suppose, to the contrary, that $G = G^*$. Let us consider edges za and zb for any $b \in B_z$. By (11), $\phi^{-1}(za) = \{z, a, x\}$ for some $x \in A \setminus \{a\}$, and $\phi^{-1}(zb) = \{z, b, y\}$ for some $y \in B \setminus B_z$. Since a is the unique vertex which is non-adjacent to both x and b , we have $\phi^{-1}(xb) = \{a, b, x\}$. The trace of $\{z, a, x\}, \{z, b, y\}, \{a, b, x\}$ on $\{z, a, b\}$ is a K_3 , contradicting $H \not\cong K_3$.

Finally we apply Lemma 4.1 and obtain that $|H| = |E| \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$. □

5. Concluding Remarks and Open Problems

A less ambitious goal than proving Conjecture 1.6 is to verify (3), or equivalently (10), for $p \geq t + 2$ and $q \leq t - 3$. This will reduce the trace problem to determining $\text{ex}(n, H_{q,t}^{t-1})$, which is only known for $q = 2$. To obtain the asymptotic value of $L^p(n, K_t^q)$ in other cases, one should try to verify (6) for $p \geq t + 2$ and $t \geq 5$; the smallest open case is to prove that

$$L^7(n, K_5) = |T_4^4(n)| + o(n^4) = \left(\frac{n}{4}\right)^4 + o(n^4).$$

Following the ideas in Sections 3.2 and 3.3, in order to extend Theorem 1.4 for all $p \geq t$, one needs to show that $G^k = o(n^{t-1})$ for $t \leq k \leq p$. When $p \geq t + 2$, this does not follow from Lemma 3.3 and the Kruskal-Katona theorem. The proof of Lemma 3.4 relies on the assumption $q = t - 2$, and does not seem to generalize to other values of q .

A general uniform trace problem is to determine $L^p(n, F)$ for arbitrary p and F . Because of the close connection between trace problems and Turán problems, as seen in Proposition 1.3 and Theorem 1.4, it is very hard to determine $L^p(n, F)$ in general. Let us consider $L^3(n, F)$ when F is a graph. Fix $t = \chi(F)$. When $t \geq 4$, we have

$$L^3(n, F) = |T_{t-1}^3(n)| + o(n^3) = \binom{t-1}{3} \left(\frac{n}{t-1}\right)^3 + o(n^3).$$

In fact, the lower bound for $L^3(n, F)$ follows from $T_{t-1}^3(n) \not\cong F$, where $T_{t-1}^3(n)$ is the generalized Turán graph defined in the introduction. The reason for $T_{t-1}^3(n) \not\cong F$ is that when embedding F into a $(t-1)$ -partite graph, some partition set must contain both ends of an edge of F . The upper bound follows from (5) and Lemma 2.4. The same arguments actually show that $L^p(n, F) = |T_{t-1}^p(n)| + o(n^{t-1})$ for every F with $t = \chi(F) > p$.

Problem 5.1. *Determine the order of magnitude of $L^3(n, F)$ for every F with $\chi(F) \leq 3$.*

This seems no easier than determining the order of magnitude of the Turán numbers for bipartite graphs. We can derive an upper bound for $L^3(n, F)$ as follows. A result of Erdős [2] implies that $\text{ex}(n, K_3^3(m)) = O(n^{3-\frac{1}{m^2}})$. For a 3-graph H , it is clear that $K_3^3(m) \subseteq H$ implies that $H \rightarrow K_3(m-1)$. For each F with $\chi(F) \leq 3$, there exists m such that $F \subseteq K_3^3(m)$. Hence $L^3(n, F) \leq L^3(n, K_3^3(m)) \leq \text{ex}(n, K_3^3(m+1)) = O(n^{3-c})$, where $c = 1/(m+1)^2$. However, we do not have a matching lower bound. For example, we only know $L^3(n, K_3(2)) = \Omega(n^{5/2})$, in contrast to the upper bound $O(n^{26/9})$ derived by above arguments (or $O(n^{11/4})$ by some extra ideas). This lower bound can be seen from the 3-partite 3-graph with partition sets A, B, C of size n , and the edge set $\{e \cup v : v \in C, e \in G\}$, where G is a maximum C_4 -free bipartite graph on (A, B) with $\Omega(n^{3/2})$ edges.

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