MATH 4010/6010: Mathematical Biology

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Lecture 2 – May 19, 2014

Linear difference method: computer lab
Non-linear Difference Methods
Matlab

- Log in
- Create a directory
- Download the m-scripts
- Run the matlab programs
- Generate and print plots
- Include the plots in a word document
- Email the results
Matlab tutorial

- Tutorial – run tutorial.m file
- Variables are assigned numerical values by typing the expression directly, for example, typing

  - $a = 1+2$
  
  - yields: $a = 3$
Matlab tutorial

- MATLAB utilizes the following arithmetic operators:
  - + addition
  - - subtraction
  - * multiplication
  - / division
  - ^ power operator
  - ' transpose
Matlab tutorial

• A variable can be assigned using a formula that utilizes these operators and either numbers or previously defined variables.

• For example, since variable $a$ was defined previously, the following expression is valid

  $b = 2*a$;
  
  Etc...
1.1 Difference equations: example 1

- Mathematical model:
  \[ M_{n+1} = a \cdot M_n \]

- \[ M_1 = a \cdot M_0; \quad M_2 = a \cdot M_1 = a \cdot (a \cdot M_0) = a^2 \cdot M_0 \]

- \[ M_3 = a \cdot M_2 = a \cdot (a^2 \cdot M_0) = a^3 \cdot M_0 \]

- In general: \[ M_n = a^n \cdot M_0 \]

- Proof by induction
1.1 Difference equations: example 1

- $|a| > 1$ – Explodes (we ignored limiting factors)
- $M_n$ increases over successive generations
- $|a| < 1$ (Extinct: $a_n \to 0$ as $n \to \infty$)
- $M_n$ decreases over successive generations,
- $a = 1$ ($a_n = 1$, equilibrium, in balance)
- $M_n$ is constant.
1.1 Difference equations: example 1

- There are five cases:
  - Divergent oscillations: $r < -1$
  - Decaying oscillations $-1 < r < 0$
  - Trivial solution $x_n = 0$: $r = 0$
  - Decaying solution: $0 < r < 1$
  - Exponential growth: $r > 1$
1.1 Difference equations: example 1

- Plot solutions for $r$ real: $r < -1$
- Use $r = -2$
- Divergent oscillations

Run `lecture3_1d`
1.1 Difference equations: example 1

- Plot solutions for \( r \) real: \(-1 < r < -1\)
- Use \( r = -0.5 \)
- Damped oscillations

Run lecture3_1c
1.1 Difference equations: example 1

- Plot solutions for $r$ real: $0 < r < 1$
- Use $r = 0.5$
- Decay

Run `lecture3_1b`
1.1 Difference equations: example 1

- Plot solutions for $r$ real: $1 < r$
- Use $r = 2$

- Exponential growth

Run lecture3_1a
1.1 Differential equations: example 2

• Consider the first order ODE: $x' = r x$.
• Solution $x = x_0 e^{rt}$.
• We plot solutions for the following parameters: $r > 0$, $r = 0$ and $r < 0$.
• Using the same values as in figure 1: $r = 2$, $r = \frac{1}{2}$, $r = -2$, $r = -1/2$.
• Notice that we don’t have anything jagged for ODE
1.1 Differential equations: example 2

- Plot solutions for $r$ real: $r < -1$
- Use $r = -2$
- Decay
- Run `lecture3_2a`
1.1 Differential equations: example 2

- Plot solutions for $r$ real: $-1 < r < -1$
- Use $r = -0.5$
- Slower decay

Run lecture3_2b
1.1 Differential equations: example 2

- Plot solutions for $r$ real: $0 < r < 1$
- Use $r = 0.5$
- Growth
- Run lecture3_2c
1.1 Differential equations: example 2

- Plot solutions for $r$ real: $1 < r$
- Use $r = 2$
- Faster growth
- Run `lecture3_2d`

![Graph showing exponential growth](attachment:image.png)
1.1 Linear difference eq – example 3

- The number of rabbits in month n is:
  \[ x_n = c_1 \phi^n + c_2 \mu^n = c_1 1.618^n + c_2 (-0.618)^n \]

- Are we done here?

- No, we still need to determine the coefficients \( c_1 \) and \( c_2 \)

- Need to use what we know about the initial conditions: \( x_0 = 1, x_1 = 1, x_2 = 2 \)
1.1 Linear difference eq – example 3

• Write down the system of equations and solve it.

\[
\begin{align*}
    x_0 &= c_1 \mu^0 + c_2 \phi^0 \\
    x_1 &= c_1 \mu^1 + c_2 \phi^1
\end{align*}
\]

\[
\begin{align*}
    1 &= c_1 \mu^0 + c_2 \phi^0 \\
    1 &= c_1 \mu^1 + c_2 \phi^1
\end{align*}
\]

• We obtain:

\[
\begin{align*}
    c_1 &= (\phi - 1) / (\phi - \mu) \approx 0.2764 \\
    c_2 &= (1 - \mu) / (\phi - \mu) \approx 0.7236
\end{align*}
\]
1.1 Linear difference eq – example 3

• We finally obtain the following formula:

\[ x_n = 0.7236 \cdot 1.618^n + 0.2736 \cdot (-0.618)^n \]

\[ x_n = \frac{1-\mu}{\phi - \mu} \cdot \phi^n + \frac{\phi - 1}{\phi - \mu} \mu^n \]
1.1 Linear difference eq – example 3

- Consider $x_{n+2} + a \ x_{n+1} + b \ x_n$
- with $a = -1$ and $b = 6$
- $x_{n+2} - x_{n+1} + 6 \ x_n = 0$
- get eigenvalues are 3 and -2.

\[ \tau = \left( -(-1) - \sqrt{1^2 - 4 \cdot (-1) \cdot 6} \right)/2 = \left( 1 - \sqrt{1^2 + 24} \right)/2 = (1 - 5)/2 = -2 \]

\[ \mu = \left( -(-1) + \sqrt{1^2 - 4 \cdot (-1) \cdot 6} \right)/2 = \left( 1 + \sqrt{1^2 + 24} \right)/2 = (1 + 5)/2 = 3 \]
1.1 Linear difference eq – example 3

- Geometric meaning of solutions is different. The full solution is all linear combinations:
  - \( x_n = c_1 (-2)^n + c_2 3^n. \)
  - Draw the two solutions.

- If \( c_1 = c_2 = 1 \) growth dominates

Run `lecture3_2d`
1.1 Linear difference eq – example 3

- $x_n = c_1 (-2)^n + c_2 3^n$.

- If $c_1 << c_2$ we see some oscillations, but the growth term eventually dominates.

Run lecture3_3
1.1 Linear difference eq – example 3

• How to find specific solutions, given initial conditions.

• Get two equations for unknowns $c_1$ and $c_2$ by plugging in for $x_0$ with $n = 0$ and $x_1$ for $n = 1$.

• Note that need exactly two terms in the series to get all the terms after it.

\[
\begin{align*}
  x_0 &= c_1 (-2)^0 + c_2 3^0 \\
  x_1 &= c_1 (-2)^1 + c_2 3^1
\end{align*}
\]
General 2\textsuperscript{nd} degree ODE – example 4

• Recall analogous ODE: \( x_{n''} + a x_{n'} + b x_n = 0 \).
• Look for solutions of the form \( x = e^{\lambda t} \).
• Leads to equation for \( \lambda \): \( \lambda^2 + a \lambda + b = 0 \).
  (characteristic poly)
• E.g. \( a = -1, b = -6 \), we get eigenvalues 3 and – 2.

• Solutions include \( x = e^{-2t}, x = e^{3t} \).
• Draw solutions as functions of time.
General 2\textsuperscript{nd} degree ODE – example 4

• $c_1 = c_2 = 1$
• $c_1 e^{-2t} + c_2 e^{3t}$
• Run lecture3_4

\[ c_1 = 1, c_2 = 1, \lambda_1 = -2, \lambda_2 = +3, \]
General 2\textsuperscript{nd} degree ODE – example 4

- \( c_1 = 50, c_2 = 1 \)
- \( c_1 e^{-2t} + c_2 e^{3t} \)
- Run lecture3_4
General 2\textsuperscript{nd} degree ODE – example 4

- Describe the full solution: all linear combinations.
- That is consequence of linearity – sums of solutions are solutions. \( c = c_1 e^{-2t} + c_2 e^{3t} \).
- How to get a specific solution: Plug in values for \( x_0 \) and \( x'_0 \); will have 2 equations in 2 unknown for \( c_1 \) and \( c_2 \).

\[
\begin{align*}
x(0) &= c_1 e^0 + c_2 e^0 \\
x(1) &= c_1 e^{-2} + c_2 e^{-3}
\end{align*}
\]
1.1 Linear difference – segmental growth

- Simplified version of this phenomenon for illustration purposes
- Prove versatility of difference equation models.
- A segmental organism grows by adding new segments at intervals of 24 hours in several possible ways
1.1 Linear difference – segmental growth

- Let us use the following notations

- \( a_n \) = number of terminal segments,
- \( b_n \) = number of next-to-terminal segments,
- \( S_n \) = total number of segments.
1.1 Linear difference – segmental growth

• The growth equations then becomes:

• $a_{n+1} = a_n \ p + 2 \ a_n \ q + b_n \ r$
• $b_{n+1} = a_n$
• $s_{n+1} = s_n + a_n \ p + 2 \ a_n \ q + b_n \ r$
1.1 Linear difference – segmental growth

- Solving for $a_n$ yields: $a_{n+1} = (p + 2q) a_n + r a_{n-1}$

- Using $p + q = 1$, it follows that the equations for $a_n$ and $b_n$ can be combined to give:

  - $a_{n+1} - (1 + q) a_n - r a_{n-1} = 0$

- Use initial conditions: $a_1 = 1$, $b_1 = 0$, so $a_0 = 0$. 
1.1 Linear difference – segmental growth

- A numerical example for $p = \frac{1}{2}$, $q = \frac{1}{2}$ and $r = \frac{1}{2}$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$a_n$</th>
<th>$b_n$</th>
<th>$s_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0000</td>
<td>0</td>
<td>1.0000</td>
</tr>
<tr>
<td>1</td>
<td>1.5000</td>
<td>1.0000</td>
<td>2.5000</td>
</tr>
<tr>
<td>2</td>
<td>2.8000</td>
<td>1.5000</td>
<td>5.3000</td>
</tr>
<tr>
<td>3</td>
<td>4.9000</td>
<td>2.8000</td>
<td>10.1000</td>
</tr>
<tr>
<td>4</td>
<td>8.7000</td>
<td>4.9000</td>
<td>18.8000</td>
</tr>
<tr>
<td>5</td>
<td>15.500</td>
<td>8.7000</td>
<td>34.3000</td>
</tr>
<tr>
<td>6</td>
<td>27.500</td>
<td>15.500</td>
<td>61.8000</td>
</tr>
<tr>
<td>7</td>
<td>49.100</td>
<td>27.500</td>
<td>110.9000</td>
</tr>
<tr>
<td>8</td>
<td>87.400</td>
<td>49.100</td>
<td>198.2000</td>
</tr>
<tr>
<td>9</td>
<td>155.600</td>
<td>87.400</td>
<td>353.8000</td>
</tr>
</tbody>
</table>
1.1 Linear difference – segmental growth

- A numerical example for $p = \frac{1}{2}$, $q = \frac{1}{2}$ and $r = \frac{1}{2}$.

- Run `lecture3_segment`
1.1 Linear difference – problem solution

- Problem 2 from page 29
- \( (a) \ X_n - 5X_{n-1} + 6x_{n-2} = 0; \ X_0 = 2, \ X_1 = 5. \)

- Consider the associated characteristic polynomial
- \( \lambda^2 - 5 \lambda + 6 = 0 \)
- Roots: \( \lambda_1 = 2, \ \lambda_2 = 3 \)
1.1 Linear difference – problem solution

- Problem 2a from page 29
- Match the initial conditions

\[
\begin{align*}
    x_0 &= c_1 \lambda_1^0 + c_2 \lambda_2^0 \\
    x_1 &= c_1 \lambda_1^1 + c_2 \lambda_2^1
\end{align*}
\]

- Obtain \(c_1 = c_2 = 1\)
- Solution: \(1 \times 2^n + 1 \times 3^n\)
1.1 Linear difference – problem solution

• (b) $X_{n+1} - 5X_n + 4X_{n-1} = 0; X_1 = 9, X_2 = 33.$

• Solution:
  
  
  $1^1^n + 8^4^n$
1.1 Linear difference – problem solution

- (c) $X_n - X_{n-2} = 0; X_1 = 3, X_2 = 5.$

- **Solution**
  
  $-1(-1)^n + 4 \times 1^n$

![Graph showing the sequence $X_n$ for $n$ ranging from 0 to 10]
1.1 Linear difference – problem solution

- (d) $X_{n+2} - 2X_{n+1} = 0; \ X_0 = 10.$

- Solution:
  
  \[
  0 \cdot 0^n + 10 \cdot 2^n
  \]
1.1 Linear difference – problem solution

- (e) $X_{n+2} + X_{n+1} - 2x_n = 0; \ X_0 = 6, \ X_1 = 3.$

- Solution:
  $3(-1)^n + 3 \times 2^n$
Linear difference equations – reading

- 1.3 SYSTEMS OF LINEAR DIFFERENCE EQUATIONS
- 1.4 LINEAR ALGEBRA REVIEW
- 1.5 WILL PLANTS BE SUCCESSFUL?
- 1.6 QUALITATIVE BEHAVIOR OF SOLUTIONS TO LINEAR DIFFERENCE EQUATIONS

- Problem 4, 6 from page 30
Love affairs example

- Strogatz introduced a model that describes the dynamical feelings between Romeo and Juliet

- Finite difference equations

- Is this a ‘biological’ system?
Love affairs example

- Assumptions
- Feelings dynamic
- Feelings depend on previous feeling states
- Feelings depend on the other’s feelings

**Q:** How does the relationship between R & J evolve?

**A:** Build a model
Love affairs example

• Step 1: Define variables
  • \( R_n = \) Romeo’s love/hate for Juliet on day \( n \)
  • \( J_n = \) Julie’s love/hate for Romeo on day \( n \)

• Interpret: (similar for \( R_n \))
  • \( J_n > 0 \), Juliet loves Romeo
  • \( J_n = 0 \), Juliet indifferent to Romeo
  • \( J_n < 0 \), Juliet hates Romeo
Love affairs example

- Step 2: Parameters
  - More difficult in this case
  - Skip to step 3
  - Write down equations that make sense

- Step 3: Relate variables and parameters
Love affairs example

- \( R_{n+1} = a_R \ R_n \)
- Romeo’s feelings depends on his feelings yesterday

- Set \( a_R > 0 \). Why?
- Otherwise Romeo’s feelings are subject to wild mood swings
- \( a_R < 0, \ R_0 > 0 \). Then \( R_1 < 0, \ R_2 > 0, \ R_3 < 0 \) etc
Love affairs example

- Consider $0 < a_R < 1$
- Solution: $R_n = (a_R)^n R_0$
- As $n \to \infty$, $R_n \to 0$: feelings neutralize over time

- Consider $1 < a_R$
- As $n \to \infty$, $R_n \to \infty$: feelings intensify over time

- Similarly for Juliet: $J_{n+1} = a_J J_n$
Love affairs example

- Now let’s add linear terms that represent the responses of Romeo and Juliet’s to each other’s feelings

- Consider:
  - $R_{n+1} = a_R R_n + p_R J_n$
  - $J_{n+1} = p_J R_n + a_J J_n$
Love affairs example

- Focus on the last term: $p_j R_n$
  - If $p_j > 0$, then $p_j R_n > 0$ when $R_n > 0$
  - Juliet is encouraged by Romeo’s love (psycho)
  - If $p_j > 0$, then $p_j R_n < 0$ when $R_n < 0$
  - Juliet is discouraged by Romeo’s hated
Love affairs example

- Focus on the last term: $p_j R_n$
  - If $p_j < 0$, then $p_j R_n > 0$ when $R_n < 0$
  - Juliet is encouraged by Romeo’s hate (psycho)
  - If $p_j < 0$, then $p_j R_n < 0$ when $R_n > 0$
  - Juliet is discouraged by Romeo’s love
Love affairs example

- Both Romeo and Juliet have four romantic styles (+, +), (-, -), (+, -), (-, +)

- The final state (outcome of the love affair) depends on the initial state and values of the interaction parameters

- Let’s express the model in matrix form
Love affairs example

- Note order of the matrix

\[
\begin{pmatrix}
R_{n+1} \\
J_{n+1}
\end{pmatrix} = \begin{pmatrix}
a_R & p_R \\
p_J & a_J
\end{pmatrix} \begin{pmatrix}
R_n \\
J_n
\end{pmatrix}
\]

\[\tilde{M} = \begin{pmatrix}
a_R & p_R \\
p_J & a_J
\end{pmatrix}\]

- Compute the eigenvalue of matrix \( M \)

\[
\det(\tilde{M} - \lambda I) = \begin{pmatrix}
a_R - \lambda & p_R \\
p_J & a_J - \lambda
\end{pmatrix}
\]
Love affairs example

- Compute the eigenvalues of matrix $M$

\[(a_R - \lambda)(a_J - \lambda) - p_Rp_J = 0\]

\[\lambda^2 - (a_R + a_J)\lambda + a_Ra_J - p_Rp_J = 0\]

\[
\lambda_1 = \frac{(a_R + a_J) - \sqrt{(a_R + a_J)^2 - 4(a_Ra_J - p_Rp_J)}}{2}
\]

\[
\lambda_2 = \frac{(a_R + a_J) + \sqrt{(a_R + a_J)^2 - 4(a_Ra_J - p_Rp_J)}}{2}
\]
Love affairs example

- We can rewrite

\[ \lambda_{1,2} = \frac{(a_R + a_J) \pm \sqrt{(a_R - a_J)^2 + 4p_R p_J}}{2} \]

- So, what’s happenin?  
- Complicated expression  
- Consider special cases
Love affairs example

- Case 1: $0 \ll p_J, p_R \ll 1$
- Interpret: Romeo and Juliet are cautious and the other’s feeling have little impact on their next state

- Therefore the term $p_J p_R$ is negligible

$$\lambda_{1,2} = \frac{(a_R + a_J) \pm \sqrt{(a_R - a_J)^2}}{2} = \frac{(a_R + a_J) \pm |a_R - a_J|}{2}$$
Love affairs example

• Case 1: 0 \ll p_J, p_R \ll 1

\lambda_{1,2} \in \{a_J, a_R\}

• If 0 < a_J, a_R < 1
• As n \to \infty, R_n \to 0, J_n \to 0, romance fizzes

• If 1 < a_J, a_R
• As n \to \infty, R_n \to \infty, J_n \to \infty, romance blossoms
Love affairs example

• Case 2:
  • $1 \ll p_J, p_R$
  • R&J encouraged by each other’s feelings
  • $0 < a_J, a_R \ll 1$
  • R&J disregard their own feelings
  • Since $1 \ll p_J p_R$

\[
\lambda_{1,2} = \frac{(a_R + a_J) \pm \sqrt{(a_R - a_J)^2 + 4p_R p_J}}{2} \approx \pm \sqrt{p_R p_J}
\]
Love affairs example

- Case 2: Numerical example
  - $p_J = 10$, $p_R = 10$
  - $a_J = 0$, $a_R = 0$

$$M = \begin{pmatrix} 0 & 10 \\ 10 & 0 \end{pmatrix}$$

$$\lambda_{1,2} = \pm \sqrt{100} = \pm 10$$

- Consider initial conditions
  - $R_0$ = \begin{pmatrix} + \varepsilon \\ - \varepsilon \end{pmatrix}$

- Romeo – initial feeling of week love
- Juliet – initial feeling of week hate
Love affairs example

- **Q:** What is the value of \( \begin{pmatrix} R_n \\ J_n \end{pmatrix} \) as \( R_n \to \infty \)?

- What happens to the relationship in the distant future?

- **A:** Run a computer simulation or just compute the first few terms.
**Love affairs example**

- **A:** Run a computer simulation

\[
\begin{pmatrix}
R_1 \\
J_1
\end{pmatrix} = \tilde{M} \begin{pmatrix}
R_0 \\
J_0
\end{pmatrix} = \begin{pmatrix}
0 & 10 \\
10 & 0
\end{pmatrix} \begin{pmatrix}
+ \varepsilon \\
- \varepsilon
\end{pmatrix} = 10 \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
+ \varepsilon \\
- \varepsilon
\end{pmatrix} = 10 \begin{pmatrix}
- \varepsilon \\
+ \varepsilon
\end{pmatrix}
\]

\[
\begin{pmatrix}
R_2 \\
J_2
\end{pmatrix} = \tilde{M}^2 \begin{pmatrix}
R_0 \\
J_0
\end{pmatrix}
\]

\[
\begin{pmatrix}
R_n \\
J_n
\end{pmatrix} = \tilde{M}^n \begin{pmatrix}
R_0 \\
J_0
\end{pmatrix}
\]
Love affairs example

- **A:** Use the eigenvector properties
- Need to compute the eigenvectors of matrix $M$
- Start with $\lambda_2 = 10$

\[
\begin{pmatrix} 0 & 10 \\ 10 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 10 \begin{pmatrix} a \\ b \end{pmatrix} \quad \Rightarrow \quad 10b = 10a \quad 10a = 10b
\]

- Resulting eigenvector is: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
Love affairs example

- **A:** Use the eigenvector properties
- Need to compute the eigenvectors of matrix $M$
- Start with $\lambda_1 = -10$

\[
\begin{pmatrix}
0 & 10 \\
10 & 0
\end{pmatrix}
\begin{pmatrix}
a \\
b
\end{pmatrix}
= -10
\begin{pmatrix}
a \\
b
\end{pmatrix}
\]

$10b = -10a$

$10a = -10b$

- Resulting eigenvector is: $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
Love affairs example

- Matlab code
  - `[evecs1, eigs1] = eigs([0 1; 1 0]);`
  - `>> diag(eigs1)`
    - 1
    - -1
  - `>> evecs1`
    - 0.7071   -0.7071
    - 0.7071   0.7071
Love affairs example

- **A:** Now express the initial condition as a combination of eigenvalues

- **Notice**

  \[
  \begin{pmatrix}
  R_0 \\
  J_0
  \end{pmatrix}
  =
  \begin{pmatrix}
  + \varepsilon \\
  - \varepsilon
  \end{pmatrix}
  =
  \varepsilon
  \begin{pmatrix}
  1 \\
  -1
  \end{pmatrix}
  = \varepsilon \tilde{\nu}_-
  \]

- **Then**

  \[
  \begin{pmatrix}
  R_n \\
  J_n
  \end{pmatrix}
  =
  \tilde{M}^n
  \begin{pmatrix}
  + \varepsilon \\
  - \varepsilon
  \end{pmatrix}
  = \varepsilon \tilde{M}^n \tilde{\nu}_-
  = \varepsilon \lambda_2^n \tilde{\nu}_-
  = \varepsilon (-10)^n \tilde{\nu}_-
  \]
Love affairs example

- **Q:** What happens as we iterate?

  - $n = 0 \quad \begin{pmatrix} R_0 \\ J_0 \end{pmatrix} = \begin{pmatrix} + \varepsilon \\ - \varepsilon \end{pmatrix} = \varepsilon \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \varepsilon \tilde{\nu}_-, \text{ weak love/hate}$

  - $n = 1 \quad \begin{pmatrix} R_1 \\ J_1 \end{pmatrix} = -10 \varepsilon \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{hate/love}$
Love affairs example

- **Q:** What happens as we iterate?

- \( n = 2 \)
  \[
  \begin{pmatrix} R_0 \\ J_0 \end{pmatrix} = 100\varepsilon \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \text{ strong love/hate}
  \]

- \( n = 3 \)
  \[
  \begin{pmatrix} R_1 \\ J_1 \end{pmatrix} = -1000\varepsilon \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \text{ stronger hate/love}
  \]
Love affairs example

- Numerical variations
  - $a_R = 0.5$;
  - $a_J = 0.7$;
  - $p_R = 0.2$;
  - $p_J = 0.5$;
- Cautious love
- fizzles
Love affairs example

- Numerical variations
  - \(a_R = 0.5\);
  - \(a_J = 0.7\);
  - \(p_R = 0.2\);
  - \(p_J = 0.5\);
Love affairs example

- Numerical variations
  - $a_R = 0.5$;
  - $a_J = 0.7$;
  - $p_R = 0.7$;
  - $p_J = 0.9$;
- Feed-back love
- It explodes
Love affairs example

- Numerical variations
  - $a_R = 1$;
  - $a_J = 1$;
  - $p_R = 0.2$;
  - $p_J = -0.2$;
- Suspicious love
- Opposite attract
Love affairs example

- Numerical variations
- $a_R = 1$;
- $a_J = 1$;
- $p_R = 0.2$;
- $p_J = -0.2$;
Love affairs example

- Numerical variations
  - $a_R = 0.5$;
  - $a_J = 0.8$;
  - $p_R = 0.2$;
  - $p_J = 0.5$;
- Equilibrium
- Perpetual love
- Unequal love
Love affairs example

- Numerical variations
- $a_R = 0.5$;
- $a_J = 0.8$;
- $p_R = 0.2$;
- $p_J = 0.5$;
Linear difference equations – reading

- 1.7 THE GOLDEN MEAN REVISITED
- 1.8 COMPLEX EIGENVALUES IN SOLUTIONS TO DIFFERENCE EQUATIONS
2. Nonlinear difference equation

- Q: What is nonlinear?
- A: $x_{n+1} = f(x_n, x_{n-1}, \ldots)$

- Ignore the past values for now
- $x_{n+1} = f(x_n)$ – depends on nonlinear terms
  - E.g. $x_n^2$, $x_n^3$, $x_n \exp(x_n)$

- What are the solutions?
Nonlinear difference equation

• **Q:** What are the solutions?
• **A:** For the linear case we know that: \( x_n = \lambda^n x_0 \)

• Nonlinear: Generally there is no analytical solution
• We need to develop other methods: fixed points, stability, bifurcations, graphical representation
• More complicated, also more interesting
Nonlinear difference equation

- **Definition:** A ‘fixed point’ (f. p.) or ‘steady state’ or ‘equilibrium’ satisfies the equation:

  \[ x_{n+1} = x_n = \bar{x} \]

  \[ \bar{x} = f(\bar{x}) \]

- Classical example: \[ x_{n+1} = r \cdot x_n (1 - x_n) \]
- Logistic equation
Nonlinear difference equation

- Classical example: $x_{n+1} = r \cdot x_n (1 - x_n)$

- f. p.: $\bar{x} = r \cdot \bar{x} (1 - \bar{x})$
  - $r \cdot \bar{x} - \bar{x} - r \cdot \bar{x}^2 = 0$
  - $\bar{x}(r - 1 - r \cdot \bar{x}) = 0$

- Solutions: $\bar{x} = 0 \quad \bar{x} = 1 - 1/r$
Nonlinear difference equation

Check:

$x_0 = 0, \quad x_1 = r x_0 (1 - x_0) = 0$
$x_2 = r x_1 (1 - x_1) = 0$

$x_0 = 1 - 1/r, \quad x_1 = r (1 - 1/r) (1 - (1 - 1/r)) = r (1 - 1/r) (1/r) = (1 - 1/r)$

$x_2 = r x_1 (1 - x_1) = (1 - 1/r)$
Nonlinear difference equation

• **Q:** Why compute steady states? Nature rarely ‘steady’?

• **A:** By examining carefully the fixed points, we can understand better the behavior of the system.

• It the system goes towards or is perturbed away from the fixed points, then we can look at the behavior around those points.
Nonlinear difference equation

• Definition: A fixed point is **stable** if neighboring states are ATTRACTED back to the fixed point.

• Definition: A fixed point is **unstable** if neighboring states are REPELLED away from the fixed point.

• Example: $x_{n+1} = r \cdot x_n (1 - x_n)$ with $r = 2$
Nonlinear difference equation

- Example: $x_{n+1} = r x_n (1 - x_n)$ with $r = 2$
- Start near $x_{bar} = 0$
Nonlinear difference equation

- Example: \( x_{n+1} = r \ x_n \ (1 - x_n) \) with \( r = 2 \)
- Start near \( x_{\text{bar}} = 1 - 1/r = 1 - \frac{1}{2} = \frac{1}{2} \)
2.2 STEADY STATES, STABILITY, AND CRITICAL PARAMETERS

• Three situations
• Two of which are steady states;
• One steady state is stable, the other unstable
2.2 STEADY STATES, STABILITY, AND CRITICAL PARAMETERS

- Imagine that the ball represents a biological population
- Near 1, a small perturbation is returned to equilibrium – system stable
- Near 2, a small perturbation induces a huge change – system unstable
- E.g. avalanches, subzero water
Analysis of stability

- Consider: \( x_{n+1} = f(x_n) \) with fixed point \( x_{\text{bar}} \). Is this f. p. stable or unstable?
- Consider what happens near the f.p.
- \( x_n = x_{\text{bar}} + x_n' \)
- \( x_n' \) is a small perturbation from \( x_{\text{bar}} \)
- **Q:** What happens? Approach \( x_{\text{bar}} \), flee \( x_{\text{bar}} \)?
- **A:** to determine, substitute in the difference equation
Analysis of stability

- $x_{n+1} = f(x_n) = f(x_{\text{bar}} + x_n')$
- Write $x_{n+1} = x_{\text{bar}} + x_{n+1}'$
- Then: $x_{\text{bar}} + x_{n+1}' = f(x_{\text{bar}} + x_n')$
- Approximate function with its Taylor expansion
- Since $x_n'$ we can ignore higher order terms
- We keep only first order term
- $x_{\text{bar}} + x_{n+1}' = f(x_{\text{bar}}) + df/dx(x_{\text{bar}}) x_n' + ...$
Analysis of stability

- Since \( x\_\text{bar} = f(x\_\text{bar}) \)
- We simplify
- \( x\_\text{bar} + x_{n+1}' = f(x\_\text{bar}) + \frac{df}{dx}(x\_\text{bar}) \cdot x_n' + ... \)
- Into
- \( x_{n+1}' = \frac{df}{dx}(x\_\text{bar}) \cdot x_n' \)
- New equation for the dynamic of the perturbation looks very similar to previous equation: denote \( a = \frac{df}{dx}(x\_\text{bar}) \)
Analysis of stability

- $x_{n+1}' = a \cdot x_n'$
- Back to linear difference equation
- Stability means that the perturbation goes back to zero. This happens if $|a| < 1$
- Unstable means that the perturbation does not go back to zero. This happens if $|a| > 1$
Analysis of stability

• Formally:

• $x_{\text{bar}}$ is a stable f.p. of $x_{n+1} = f(x_n)$ when $|df/dx(x_{\text{bar}})| < 1$

• $x_{\text{bar}}$ is an unstable f.p. of $x_{n+1} = f(x_n)$ when $|df/dx(x_{\text{bar}})| > 1$
Analysis of stability

- If \[ \left| \frac{df}{dx} \right|_{\bar{x}} = 1 \]

- We just don’t know stability.
- We call these f.p.s ‘non-hyperbolic’

- We need additional calculations to determine stability: think second order
Analysis of stability

- Example: $x_{n+1} = r \times x_n \times (1 - x_n)$

- Hence: $f(x) = r \times x \times (1 - x)$

- Evaluate the derivative at the fixed point

$$\left| {\frac{df}{dx}} \right|_{x_{\text{fixed}}} = r(1 - 2x) = r(1 - 2(1 - 1/r)) = r(-1 + 2/r) = 2 - r$$
Analysis of stability

• Stability of the fixed point depends on the parameter $r$ ($a = 2 - r$)

\[
\left( \frac{df}{dx} \right)_{x=\bar{x}} = r(1 - 2x) = r(1 - 2(1 - 1/r)) = r(-1 + 2/r) = 2 - r
\]

• $r = 2.5$

Initial state $x_0 = 0.7$
Analysis of stability

- $a = 2 - r$
- Then $r = 4$ is not leading to a stable f.p.?
Analysis of stability

- Another example: nonlinear difference equation for population growth

\[ x_{n+1} = \frac{kx_n}{b + x_n} \quad b, \ k > 0 \]

- Compute the steady state, or fixed points

\[ \bar{x} = \frac{k\bar{x}}{b + \bar{x}} \]

\[ \bar{x}(b - k + \bar{x}) = 0 \]
Analysis of stability

• Fixed points \( \bar{x} = 0 \) OR \( \bar{x} = k - b \)

• Second steady state makes sense only if \( k > b \)

• Compute the first derivative

\[
f(x) = \frac{kx}{b + x} \quad \frac{df}{dx} = \frac{k(b + x) - kx}{(b + x)^2} = \frac{kb}{(b + x)^2}
\]
Analysis of stability

- Fixed points \( \bar{x} = 0 \) OR \( \bar{x} = k - b \)
- Evaluate the first derivative in the fixed points
  
  \[
  \left. \frac{df}{dx} \right|_{\bar{x}=0} = \frac{kb}{(b+0)^2} = \frac{k}{b}
  \]
  
  \[
  \left. \frac{df}{dx} \right|_{\bar{x}=k-b} = \frac{kb}{(b+k-b)^2} = \frac{kb}{k^2} = \frac{b}{k}
  \]

- They can’t be both be stable
Analysis of stability

- $k = 2$, $b = 1$
- Fixed points $x_{\text{bar}} = 0$, or $x_{\text{bar}} = k - b = 1$

- $\frac{df}{dx}(x_{\text{bar}} = 0) = k/b = 2$
  - Unstable

- $\frac{df}{dx}(x_{\text{bar}} = 1) = b/k = \frac{1}{2}$
  - Stable
Analysis of stability

- k = 1, b = 2
- Fixed points x_bar = 0, or x_bar = k – b = -1 (Non-biological)

  \[ \frac{df}{dx}(x_{\text{bar}} = 0) = \frac{k}{b} = \frac{1}{2} \]
  - Stable

  \[ \frac{df}{dx}(x_{\text{bar}} = -1) = \frac{b}{k} = 2 \]
  - Unstable
Analysis of stability

- $k = 1$, $b = 1$
- Fixed points $x_{\text{bar}} = 0$, or $x_{\text{bar}} = k - b = 0$

- $\frac{df}{dx}(x_{\text{bar}} = 0) = \frac{k}{b} = 1$
- $\frac{df}{dx}(x_{\text{bar}} = 0) = \frac{b}{k} = 1$

- First order analysis is inadequate
- Numerically is stable?
Graphical analysis

- $k = 2, b = 1$
- Tan at 1\text{st} point $> 1$
- Unstable
- Tan at 2\text{nd} point $< 1$
- Stable
- $f$ saturates at $k/b$
- 2\text{nd} intersect at $k - b$

$f(x) = k \frac{x}{b + x}; \ g(x) = x$, for $k = 2, b = 1$
Graphical analysis

- \( k = 1, \ b = 1 \)
- \( \tan \) at 1\(^{st}\) point < 1
- Stable
- No second intersections

\[ f(x) = \frac{k \ x}{b + x}; \quad g(x) = x, \text{ for } k = 1, \ b = 1 \]
Graphical analysis

- \( k = 1, \ b = 2 \)
- \(|\tan|\text{ at } 1^{\text{st}} \text{ point} > 1\)
- Unstable
- Non-biological

- \(\tan\text{ at } 2^{\text{nd}} \text{ point} < 1\)
- Stable

\[ f(x) = \frac{k \ x}{b + x}; \ g(x) = x, \text{ for } k = 1, \ b = 2 \]
Graphical analysis

- Something important happens at $k/b = 1$
- Transition from one behavior to another
2.3 The Logistic Difference Equation

• Classic paper by May (1976)

• \( x_{n+1} = r x_n (1 - x_n) \)

• Two regimes

• \( x_n << 1 \), then \( x_{n+1} \approx r x_n \) (rate proportional to the current population)

• \( 0 << x_n < 1 \), then \( x_{n+1} \approx r (1 - x_n) \) (1 represents the theoretical "carrying capacity")
2.3 The Logistic Difference Equation

- Equivalent formulation:
- $y_{n+1} = y_n(r - d \cdot y_n)$

- $y_{n+1} = r \cdot y_n (1 - d/r \cdot y_n)$
- Redefine $x_n = d/r \cdot y_n$
- *Substitute and obtain initial equation*
- *Useful to think the real number of parameters*
Restrict analysis to the following ranges

- \[0 < x < 1, \ 1 < r < 4\]
- otherwise the population becomes extinct

- We’ll look at the behavior of this simple system in detail
- But before that, we’ll see what is the behavior of continuous system.
- This will be in stark contrast to the discrete case.
2.3 The Logistic Difference Equation

- *Pearl-Verhulst or logistic equation,*
- Describe continuous density-dependent growth rates of populations.
2.3 The Logistic Difference Equation

- Classical example: $x_{n+1} = r x_n (1 - x_n)$

- Fixed points: $\bar{x} = r\bar{x}(1 - \bar{x})$

- Solutions:
  $\bar{x} = 0$
  $\bar{x} = 1 - \frac{1}{r}$

\[
\begin{align*}
\left| \left( \frac{df}{dx} \right) \right|_{\bar{x}} &= r(1 - 2\bar{x}) \\
\left| \left( \frac{df}{dx} \right) \right|_0 &= r \\
\left| \left( \frac{df}{dx} \right) \right|_{1 - \frac{1}{r}} &= 2 - r
\end{align*}
\]
2.3 The Logistic Difference Equation

- If $0 < r < 1$, first f.p. is stable
- the second is unstable

\[
\left\| \left( \frac{df}{dx} \right) \right\|_0 = r
\]

\[
\left\| \left( \frac{df}{dx} \right) \right\|_{1-\frac{1}{r}} = 2 - r
\]

$r = 1.2, x_0 = 0.26667, \text{f.p.} = 0 \text{ and } 0.16667$
2.3 The Logistic Difference Equation

- If $1 < r < 3$
- first f.p. is unstable
- the second is stable

\[
\left| \left( \frac{df}{dx} \right) \right|_0 = r
\]

\[
\left| \left( \frac{df}{dx} \right) \right|_{1-\frac{1}{r}} = 2 - r
\]
2.3 The Logistic Difference Equation

- If $1 < r < 3$
- first f.p. is unstable
- the second is stable

\[
\left| \frac{df}{dx} \right|_0 = r \\
\left| \frac{df}{dx} \right|_{1 - \frac{1}{r}} = 2 - r
\]
2.3 The Logistic Difference Equation

- Something happens at when $r > 3$
- Both fixed points are unstable

$r = 3.1, x_0 = 0.77742, f.p. = 0$ and $0.67742$

$r = 4, x_0 = 0.85, f.p. = 0$ and $0.75$
2.4 BEYOND $r = 3$

- Q: what happens when $r > 3$?
- A: numerics, we see oscillations of period 2, that is, values alternate between say $x_1$ and $x_2$

- Q: Stable behavior?
- A: Yes, at least numerically
2.4 BEYOND $r = 3$

- Q: Are oscillations observed in general?
- A: No, other type of dynamics were numerically observed.

- Q: Stable behavior?
- A: We don’t know.
2.4 BEYOND $r = 3$

- Q: Can we do any math about the oscillations of period 2
- A: Yes, define a period 2 oscillation, between $\bar{x}_1$ and $\bar{x}_2$

- $x_{n+1} = f(x_n)$
- $x_{n+2} = x_n$

$r = 3.1, x_0 = 0.77742, f.p. = 0$ and $0.67742$
2.4 BEYOND $r = 3$

- $x_{n+1} = f(x_n) = f(f(x_{n-1})) = g(x_{n-1})$

- Now find out the fixed points of equation
  
  \[ \bar{x}_1 = g(\bar{x}_1) \quad \bar{x}_2 = g(\bar{x}_2) \]

- $y = r \times (1 - x)$
- $r \times y (1 - y) = r \times (1 - x) (1 - r \times (1 - x))$
- $g(x) = r \times (1 - x) (1 - r \times (1 - x))$
2.4 BEYOND $r = 3$

- $x = g(x) = r x (1 - x) (1 - r x (1 - x))$
- We mean $x_{\text{bar}}$ instead of $x$ here

- $x = r x (1 - x) (1 - r x (1 - x))$

- Trivial solution is still here: $x = 0$
- $r (1 - x) (1 - r x (1 - x)) = 1$; third degree poly
2.4 BEYOND $r = 3$

- $r \ (1 - x) \ (1 - r \ x \ (1 - x)) = 1$; third degree poly
- In general very complicated to solve

- BUT: we know that $x = 1 - 1/r$ MUST be a period 2 fixed point!!!

- So we can factor out that solution, and obtain a second degree equation for the other two roots
2.4 BEYOND \( r = 3 \)

- After equation manipulations (or just using Mathematica) we obtain
  \[
  \bar{x}_{1,2} = \frac{1 + r \pm \sqrt{(r - 3)(r + 1)}}{2}
  \]
- These solutions exist when \( r > 3 \)
- They come into existence just as the f.p.
  \( \bar{x} = 1 - 1/r \) becomes unstable
2.4 BEYOND $r = 3$

- Now we need to determine their stability
- Evaluate $g'(x)$
- Mathematica: $-r^2 (-1+2 x) (1+2 r (-1+x) x)$

- Evaluate the derivative in the 4 fixed points:
  - $r^2$, $(-2+r)^2$, $4+2 r-r^2$, $4+2 r-r^2$
  - The roots of $4+2 r-r^2$ are $1 - \sqrt{5}$ and $1 + \sqrt{5}$
2.4 BEYOND r = 3

- The roots of $4+2 r-r^2$ are approx -1.2361 and 3.2361

- Alternatively (see problem 5 chapter 2)

\[ x_i \text{ is a stable 2-point cycle } \Leftrightarrow \left| \left( \frac{df}{dx} \big|_{x_1} \right) \left( \frac{df}{dx} \big|_{x_2} \right) \right| < 1 \]
2.4 BEYOND $r = 3$

- The question can now be asked, what happens if $r$ exceeds the critical value and the period 2 solutions become unstable?

- We can attempt to use the same methods for period 3 or period 4 solutions, etc.

- Algebra becomes quite complicated

- Graphical methods
2.4 BEYOND $r = 3$

- Suffice to say: classical example of bifurcation and chaos
2.4 BEYOND $r = 3$

- System undergoes multiple bifurcations of stable multiple period solutions
- Then it reaches a state where small changes in initial conditions lead to large variation in the iterated states
2.5 Graphical Methods 4 first-order Eqs

- Q: Why choose graphical methods?
- A:
  1. Visualize stability and fixed points
  2. Observe how f.p. change as parameters are varied (bifurcations)
  3. No ‘math’
2.5 Graphical Methods 4 first-order Eqs

- Logistic equations
- We can learn a lot from the graphs about the changes induced by the parameter \( r \)
- Consider \( 0 < r < 1 \)
2.5 Graphical Methods 4 first-order Eqs

• Consider $1 < r < 3$; Select $r = 1.5$

$f(x) = r x (1 + x)$; $g(x) = x$, for $r = 1.5$

$r = 1.5$, $x_0 = 0.43333$, f.p. = 0 and 0.33333
2.5 Graphical Methods 4 first-order Eqs

- Consider $1 < r < 3$;
- Select $r = 2$

\[
f(x) = r x (1 + x); \quad g(x) = x, \text{ for } r = 2
\]

\[
r = 2, \ x_0 = 0.6, \ f.p. = 0 \text{ and } 0.5
\]
2.5 Graphical Methods 4 first-order Eqs

- Consider $1 < r < 3$; Select $r = 2.5$

$f(x) = r \times (1 + x); g(x) = x$, for $r = 2.5$

$r = 2.5, x_0 = 0.7$, f.p. = 0 and 0.6
2.5 Graphical Methods 4 first-order Eqs

- Choose $x_0 = 0.7$ on the left graph

\[
f(x) = r x (1 + x); \quad g(x) = x, \text{ for } r = 2.5
\]

\[
r = 2.5, \ x_0 = 0.7, \text{ f.p. } = 0 \text{ and } 0.6
\]
2.5 Graphical Methods 4 first-order Eqs

- Choose $x_0 = 0.7$
2.5 Graphical Methods for First-Order Eqs

- Compute $f(x_0)$

Graphical methods are used to visualize and solve first-order differential equations. The graph shows the function $f(x) = rx(1+x)$, where $g(x) = x$, for $r = 2.5$. The initial condition is $x_0 = 0.7$, and the next approximation is $x_1 = f(x_0) = 0.52$. The diagram illustrates the iterative process used in graphical methods.
2.5 Graphical Methods 4 first-order Eqs

- Start again at \( x_1 \) this time
2.5 Graphical Methods 4 first-order Eqs

• Compute $x_2 = f(x_1)$
2.5 Graphical Methods 4 first-order Eqs

- But we can do this directly on the graph

\[ f(x) = r \times (1 + x); \quad g(x) = x, \text{ for } r = 2.5 \]

Choose \( x_0 = 0.7 \)

Compute \( x_2 = f(x_1) \)

\( x_2 = 0.62 \)

Shortcut
2.5 Graphical Methods 4 first-order Eqs

- Hence outlining a procedure for iterations

\[ f(x) = r \times (1 + x); g(x) = x, \text{ for } r = 2.5 \]
2.5 Graphical Methods 4 first-order Eqs

• ‘Coweb’ procedure for iterations:
  • Steps to iterate:
  • 1) ‘up/down to curve f(x)’
  • 2) over to the diagonal
  • Repeat

• Continue iterations until?
• Reach a fixed point.
2.5 Graphical Methods 4 first-order Eqs

- Intersections are the fixed points

- Furthermore Stability
2.5 Graphical Methods 4 first-order Eqs

- Use coweb to check stability of the fixed points.
- For stability we need
  \[ \left| \left( \frac{df}{dx} \right)_{x=x_0} \right| < 1 \]

- Graphically this means that the tangent line of \( f(x) \) at the fixed point has a slope no steeper than 1 (between -45 and 45 degrees from horizontal)
2.5 Graphical Methods 4 first-order Eqs

- Stable if in between -45 and 45 degrees

\[
\left| \left( \frac{df}{dx} \right) \right|_{x=\bar{x}} < 1
\]
2.5 Graphical Methods 4 first-order Eqs

- To recap
2.5 Graphical Methods 4 first-order Eqs

• To recap
2.5 Graphical Methods 4 first-order Eqs

- Let’s revisit period doubling and plot $f(f(x))$ as parameter $r$ increases: $r = 1, r = 2.5$

\[ f(f(x)) = r^2 x (1 - x) (1 - r x (1 - x)); \quad g(x) = x, \text{ for } r = 1 \]

\[ f(f(x)) = r^2 x (1 - x) (1 - r x (1 - x)); \quad g(x) = x, \text{ for } r = 2.5 \]
2.5 Graphical Methods 4 first-order Eqs

- A transition occurs at $r = 3$

\[
f(f(x)) = r^2 x (1 - x) (1 - r x (1 - x)); g(x) = x, \text{ for } r = 3
\]

\[
f(f(x)) = r^2 x (1 - x) (1 - r x (1 - x)); g(x) = x, \text{ for } r = 3.2
\]
2.5 Graphical Methods 4 first-order Eqs

- Notice formation of 4 fixed points above $r = 3$

$$f(f(x)) = r^2 x (1 - x) (1 - r x (1 - x)); \ g(x) = x, \text{ for } r = 3.2$$
2.5 Graphical Methods 4 first-order Eqs

- Then the period doubling becomes unstable

\[ f(f(x)) = r^2 x (1 - x) (1 - r x (1 - x)); \quad g(x) = x, \text{ for } r = 3.4 \]
2.5 Graphical Methods 4 first-order Eqs

• To summarize

\[ f(f(x)) = r^2 x (1 - x) (1 - r x (1 - x)); \quad g(x) = x, \text{ for } r = 3.4 \]
2.5 Graphical Methods 4 first-order Eqs

• As mentioned we can go on like this ...
• But we’ll stop because of exponential growth :)

\[ f(f(f(f(x)))); \ g(x) = x, \text{ for } r = 3.2 \]

\[ f(f(f(f(x)))); \ g(x) = x, \text{ for } r = 3.5 \]
2.5 Graphical Methods 4 first-order Eqs

- Another example $x_n = f(x_{n-1}) = r x_{n-1}^3$

- 3 fixed points

- U, S, U
2.5 Graphical Methods 4 first-order Eqs

• Numerical check for stability of $x_{\text{bar}} = 0$

$f(x) = r x^3$, $sr = 1$, $x_0 = 0.5$, f.p. = 0 and 0
2.5 Graphical Methods 4 first-order Eqs

- Another cobweb example
- $x_{n+1} = -x_n$
- $f(x) = -x$

- Fixed points? Stability?
Reading

• 2.3 THE LOGISTIC DIFFERENCE EQUATION
• 2.4 BEYOND $r = 3$
• 2.5 GRAPHICAL METHODS FOR FIRST-ORDER EQUATIONS

• Problem 1 from page 61