Homeomorphically Irreducible Spanning Trees

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Abstract. We show that if $G$ is a graph such that every edge is in at least two triangles, then $G$ contains a spanning tree with no vertex of degree 2 (a homeomorphically irreducible spanning tree). This result was originally asked in a question format by Albertson, Berman, Hutchinson, and Thomassen in 1979, and then conjectured to be true by Archdeacon in 2009.

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1 Introduction

In this paper, we consider simple and finite graphs only and assume that all graphs are connected, and refer to Bondy and Murty [BM08] for notations and terminologies used but not defined here. A graph is called homeomorphically irreducible if it does not contain any vertices of degree 2. A homeomorphically irreducible tree is called a HIT, and a homeomorphically irreducible spanning tree of a graph is called a HIST of the graph. Since every connected graph of order 3 contains a vertex of degree 2, we conventionally assume in this paper that every graph has at least 4 vertices unless we specifically name a graph with 3 vertices.

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The counterpart notions of HIT and HIST are path, cycle, and hamiltonicity, in which all vertices have degree 2 with two possible exceptions. The hamiltonian problem has been one of few fundamental problems in graph theory. However, the problem of deciding whether a graph contains a HIST has relatively short history. Harary and Prins [HP59] and Read [Rea82] enumerated both labeled and unlabeled HITs. In 1974, Hill [Hil74] studied HISTs and conjectured that every triangulation of the plane contains a HIST. In his Ph.D. dissertation, Joffe [Jof82] vigorously investigated HISTs. In 1974, Hill [Hil74] studied HISTs and conjectured that every triangulation of the plane contains a HIST. In his Ph.D. dissertation, Joffe [Jof82] vigorously investigated HISTs. In 1979, Malkevitch [Mal79] made a stronger conjecture that if G is a plane graph such that every face is a triangle with one possible exception, then G contains a HIST. In 1990, Albertson, Berman, Hutchinson, and Thomassen [ABHT90] confirmed the above conjecture, and asked whether every triangulation of a surface contains a HIST. More generally, they asked whether a graph with every edge in at least two triangles contains a HIST. To establish a strategy to tackle the problem, Ellingham [Ell96] asked whether every triangulation of a given surface with sufficiently large representativity contains a HIST. Huneke observed that every triangulation of the projective plane contains a spanning plane subgraph such that every face is a triangle with one possible exception, so every triangulation of the projective plane contains a HIST. Davidow, Hutchinson, and Huneke [DHH95] showed that every triangulation of the torus contains a HIST. In 2009, Achdeacon [BW09] (Chapter 15) restated the above two questions as two conjectures. Nakamoto and Tsuchiya [NT11] recently answered Ellingham’s question positively. With Ren, in [CRS12], we proved that every connected and locally connected graph contains a HIST. Consequently, every triangulation of a surface contains a HIST since every triangulation of a surface is connected and locally connected (Ringel [Rin78]). In this paper, we answer the second question raised by Albertson et al. positively as follows, whose proof will be given in the next section. We would like to mention that the main proof technique used in the proof is similar to that in [CRS12]. However, the induction proceeded on the spanning Θ−patch graph H (we will define it in the following section) of G is not straightforward. In fact, when H has property $Q_2$ (defined in section 2), we can not directly proceed the induction. The
new approach in dealing with this case, looks easy and natural, yet really
took efforts to come out.

**Theorem 1.** Let $G$ be a graph with every edge in at least two triangles.
Then $G$ contains a HIST.

## 2 Proof of Theorem 1

The proof consists of three main components: (1) define a class of graphs
called $\Theta$-patch graphs (we will define this class of graphs very shortly), and
show that every graph with each edge in at least two triangles contains a
spanning $\Theta$-patch graph; (2) prove a rearrangeability of $\Theta$-patch graphs; and
(3) show every $\Theta$-patch graph contains a HIST. Throughout this section, a
graph isomorphic to $K_4^-$ ($K_4$ with exactly one edge removed) is called a
$\Theta$-graph.

**Definition 2.1.** Given a graph $H$ and a vertex $v \notin V(H)$, let $H \Delta v$ be a
graph with $V(H \Delta v) = V(H) \cup \{v\}$ and $E(H \Delta v) = E(H) \cup \{u_1v, u_2v, u_1u_2\}$,
where $u_1, u_2 \in V(H)$ are two distinct vertices. That is, $H \Delta v$ is obtained
from $H$ by adding a new vertex $v$ and edges $u_1v, u_2v,$ and $u_1u_2$ if $u_1u_2 \notin
E(H)$. We name such an operation $\Delta$-operation and denote by $A(v) :=
\{u_1, u_2\}$, the set of attachments of $v$ on $H$. Moreover, we let $A[v] := A(v) \cup
\{v\}$.

Note that $u_1u_2$ may or may not be an edge of $H$.

**Definition 2.2.** Given a graph $H$ and a $\Theta$-graph $F$ with a specified degree
3 vertex, let $H \Theta F$ be the graph obtained by identifying the specified vertex
of $F$ with a vertex $u$ in $H$. Let $A(F) = \{u\}$ be the set of the attachment of
$F$ on $H$. Such an operation is called a $\Theta$-operation.

We use $\oplus$ to denote either a $\Delta$-operation or a $\Theta$-operation.

**Definition 2.3.** A graph $G$ is called a $\Theta$-patch graph if there exists a sub-
graph sequence $G_1 \subset G_2 \subset \cdots \subset G_s = G$ with $s \geq 2$ such that
Lemma 2.1. A connected graph with every edge in at least two triangles contains a \( \Theta \)-patch graph as a spanning subgraph.

Proof. Let \( G \) be a graph such that every edge is in at least two triangles. Since two triangles sharing a common edge induce a \( \Theta \)-graph, \( G \) contains a \( \Theta \)-graph, which is also a \( \Theta \)-patch graph by Definition 2.3. Let \( H \subset G \) be a \( \Theta \)-patch graph such that \( |V(H)| \) is maximum. If \( V(H) = V(G) \), the proof is completed. So assume the contrary: \( W = V(G) - V(H) \neq \emptyset \).

Since \( G \) is connected, there is an edge \( uw \in E(G) \) such that \( u \in V(H) \) and \( w \in W \). Let \( v_1uwv_1 \) and \( v_2uwv_2 \) be two distinct triangles containing \( uw \). If \( v_i \in V(H) \) for some \( i = 1, 2 \), then \( H \Delta w \) with \( A(w) = \{u, v_i\} \) is a \( \Theta \)-patch graph larger than \( H \), contradicting the maximality of \( H \). Hence, we have both \( v_1, v_2 \in W \). Clearly, \( G[\{u, v_1, v_2, w\}] \), the subgraph induced on \( \{u, v_1, v_2, w\} \), contains a \( \Theta \)-graph \( F \). So \( H\Theta F \) with \( A(F) = \{u\} \) is a \( \Theta \)-patch graph larger than \( H \), contradicting the maximality of \( H \). \( \Box \)

It will be shown in the following lemma that the ordering of subgraph
sequence in the definition of Θ-patch graphs can be rearranged to preserve a nice recursive property.

**Lemma 2.2.** Let $G$ be a Θ-patch graph of order $n \geq 5$. Then there exist a subgraph $H$ which is either a Θ-patch graph or isomorphic to $K_3$ such that one of the following properties holds:

- $P$ : $G = (H \Delta x_1) \Delta x_2$ and $A(x_2) \cap A[x_1] \neq \emptyset$;
- $Q_k$ ($0 \leq k \leq 3$) : There exist vertices $x_1, x_2, \ldots, x_k$ such that $G = (H \Theta F) \Delta x_1 \Delta x_2 \cdots \Delta x_k$ ($G = H \Theta F$ when $k = 0$) with $A[x_i] \cap A[x_j] = \emptyset$ for all $i \neq j$ and $A(x_i) \cap (V(F) - V(H)) \neq \emptyset$ for $i = 1, 2, \ldots, k$.

**Proof.** If $n = 5$, from the definition of Θ-patch graphs, there exist two vertices $x_1$ and $x_2$ such that $G = K_3 \Delta x_1 \Delta x_2$ and $A(x_2) \cap A[x_1] \neq \emptyset$, so $P$ holds. We assume that $n \geq 6$ and Lemma 2.2 holds for graphs with order $< n$.

By the definition of Θ-patch graphs, $G = H^* \oplus F^*$, where $H^*$ is a Θ-patch graph, and $F^*$ is either a single vertex or a Θ-graph. If $F^*$ is a Θ-graph, then $Q_0$ holds. So, we assume $F^*$ is a single vertex graph, and say $V(F^*) = \{w\}$. By applying Lemma 2.2 to $H^*$, we divide the remaining proof into two cases below.

**Case P.** $H^* = (H \Delta x_1) \Delta x_2$ and $A(x_2) \cap A[x_1] \neq \emptyset$.

If $A(w) \cap \{x_1, x_2\} = \emptyset$, let $H' := H \Delta w$, which is a Θ-patch graph and a subgraph of $G$. Then $G = (H' \Delta x_1) \Delta x_2$, so $P$ holds.

Suppose $A(w) \cap \{x_1, x_2\} \neq \emptyset$. If $x_1 \in A(x_2)$ or $x_2 \in A(w)$, $H' := H \Delta x_1 \subset G$ is a Θ-patch graph. Then, we have $G = (H' \Delta x_2) \Delta w$ and either $x_1 \in A(w) \cap A[x_2]$ or $x_2 \in A(w)$, so $P$ holds. We may assume that $x_1 \notin A(x_2)$ and $x_2 \notin A(w)$. In this case, we have $x_1 \in A(w)$. Let $H' := H \Delta x_2$, which is a Θ-patch graph and a subgraph of $G$. Then $G = H' \Delta x_1 \Delta w$, so $P$ holds.
Case $Q_k$. $H^* = (H\Theta F)\Delta x_1\Delta x_2 \cdots \Delta x_k$, where $F$ is a $\Theta$-graph and $x_i$ is a vertex in $H^*$.

If $A(w) \cap ((V(F) - V(H)) \cup \{x_1, x_2, \cdots, x_k\}) = \emptyset$, then

$$G = ((H\Delta w)\Theta F)\Delta x_1\Delta x_2 \cdots \Delta x_k,$$

so $Q_k$ holds. If $A(w) \cap A[x_i] \neq \emptyset$, w.l.o.g., say $A(w) \cap A[x_k] \neq \emptyset$, then

$$G = (H\Theta F\Delta x_1\Delta x_2 \cdots \Delta x_{k-1})\Delta x_k\Delta w,$$

so $P$ holds. Hence, we assume $A(w) \cap (V(F) - V(H)) \neq \emptyset$ and $A(w) \cap A[x_i] = \emptyset$ for $i = 1, 2, \cdots, k$. Under this assumption together with the assumption that $A[x_i] \cap A[x_j] = \emptyset$ for $i \neq j$ and $A(x_i) \cap (V(F) - V(H)) \neq \emptyset$ for $i = 1, 2, \cdots, k$, we have $k \leq 2$. Then, we have

$$G = (H\Theta F)\Delta x_1\Delta x_2 \cdots \Delta x_k\Delta w,$$

so $Q_{k+1}$ holds. \hfill \Box

Lemma 2.3. Every $\Theta$-patch graph contains a HIST.

Proof. We use induction on $n = |V(G)|$. When $n = 4$, $G \cong K_4^-$ is a $\Theta$-graph. Clearly, $G$ contains a HIST. Suppose $n \geq 5$, and assume that Lemma 2.3 holds for graphs of order $< n$. We divide the remaining proof into five cases according to the five properties given in Lemma 2.2.

If $G$ has property $Q_i$ for some $i = 0, 1, 2$ or 3, we follow the notations given in Lemma 2.2, and assume that $A(F) = \{u\}$ and $V(F) - V(H) = \{v_1, v_2, v_3\}$. If $G$ has property $P$ then $u$ is a specially selected vertex in $H$. We let $T$ be a HIST of $H$ if $H$ is a $\Theta$-patch graph, and let $T \cong P_3$ with $d_T(u) = 2$ if $H \cong K_3$. The case that $G$ satisfies property $Q_2$ is the most complicated one, and we can not straightforwardly play induction on it, so we defer this case to the end.

Property $P$ holds. Suppose that $G = H\Delta x_1\Delta x_2$ and $A(x_2) \cap A[x_1] \neq \emptyset$.

In this case, we first show that $N(x_1) \cap N(x_2) \cap V(H) \neq \emptyset$. This is clearly true if $A(x_1) \cap A(x_2) \neq \emptyset$, so we may assume $x_1 \in A(x_2)$. Let $u$ be
the other vertex in $A(x_2)$. Since $E(G) = E(H \Delta x_1) \Delta x_2 = E(H \Delta x_1) \cup \{x_2u, x_2x_1, ux_1\}$, we have $ux_1 \in E(G)$, that is, $u \in N(x_1) \cap N(x_2)$.

Let $u \in N(x_1) \cap N(x_2)$. Then, it is readily seen that $T \cup \{ux_1, ux_2\}$ is a HIST of $G$.

**Property Q$_0$ holds.** Let $G = H \Theta F$.

In this case, $T \cup \{uv_1, uv_2, uv_3\}$ is a HIST of $G$.

**Property Q$_1$ holds.** Let $G = (H \Theta F) \Delta x_1$, and assume, without loss of generality, $v_1 \in A(x_1) \cap (V(F) - V(H))$, and let $w_1$ be another vertex of $A(x_1)$.

In this case, $T \cup \{w_1v_1, w_1x_1, uv_1, ux_1, uv_3\}$ is a HIST of $G$ regardless of whether $w_1 \in V(F)$ or not.

**Property Q$_3$ holds.** Let $G = (H \Theta F) \Delta x_1 \Delta x_2 \Delta x_3$ and assume that $A(x_i) = \{v_i, w_i\}$ for each $i = 1, 2, 3$ with $w_1, w_2, w_3 \in V(H)$.

By the definition of $\Delta$-operation, all three edges $w_1v_1, w_2v_2, w_3v_3$ are in $E(G)$. Then, $T \cup \{w_1x_1, w_1v_1, w_2x_2, w_2v_2, w_3x_3, w_3v_3\}$ is a HIST in $G$.

**Property Q$_2$ holds.** Let $G = (H \Theta F) \Delta x_1 \Delta x_2$ such that $A(x_i) \cap (V(F) - V(H)) \neq \emptyset$ for each $i = 1, 2$, and $A[x_2] \cap A[x_1] = \emptyset$. Assume that $A(x_i) = \{v_i, w_i\}$ for $i = 1, 2$.

We may assume $w_i \neq u$ for each $i = 1, 2$; otherwise, say $w_1 = u$, then $T \cup \{w_2v_2, w_2x_2, uv_1, ux_1, w_3\}$ is a HIST of $G$. Since $A[x_2] \cap A[x_1] = \emptyset$, we may assume that $w_1 \in V(H) - \{u\}$. Moreover, under the assumption that $w_1 \in V(H) - \{u\}$, let notation be chosen so that $v_1$ is the degree 2 vertex in $F - u$ whenever it is possible, that is, if $w_2 \in V(H) - \{u\}$ and $v_2$ is the degree two vertex in $F - u$, we rename $x_2, v_2$ and $w_2$ as $x_1, v_1$ and $w_1$, and vice versa.
Let \( z \notin V(G) \) be a vertex and \( G' := H \Delta z \) with \( A(z) = \{u, w_1\} \). Clearly, \( uw_1 \in E(G') \) although \( uw_1 \) may not be in \( E(G) \). Clearly, \( G' \) is a \( \Theta \)-patch graph and \( |V(G')| < n \), so it contains a HIST \( T' \). Since \( d_{G'}(z) = 2 \), \( z \) is a degree 1 vertex of \( T' \). So, we have either \( w_1z \in E(T') \) or \( uz \in E(T') \) but not both. Let \( T_H := T' - z \).

**Subcase 1.** \( uw_1 \notin E(T') \) or \( uw_1 \in E(T') \cap E(G) \).

Note that \( d_{T'}(z) = 1 \). If \( uz \in E(T') \), let \( T^* := T_H \cup \{uv_3, w_1v_1, w_1x_1, w_2v_2, w_2x_2\} \), as depicted in Figure 2. It is routine to check that \( T^* \) is a spanning tree of \( G \) and the following equalities/inequalities hold.

\[
\begin{align*}
d_{T^*}(u) & = d_{T'}(u) - |\{uz\}| + |\{uv_3\}| = d_{T'}(u) - 2 \\
d_{T^*}(w_1) & = d_{T'}(w_1) + |\{w_1v_1, w_1x_2\}| = d_{T'}(w_1) + 2 \\
d_{T^*}(w_2) & = \begin{cases} d_{T'}(w_2) + |\{w_2v_2, w_2x_2\}| = d_{T'}(w_2) + 2 \neq 2, & \text{if } w_2 \in V(H); \\ |\{w_2v_2, w_2x_2, uv_3\}| = 3, & \text{if } w_2 = v_3. \end{cases} \\
d_{T^*}(x) & = d_{T'}(x) \neq 2 \quad \text{for all other vertices } x \in V(H), \text{ and} \\
d_{T^*}(x) & \neq 2 \quad \text{for each vertex } x \in \{v_1, v_2, v_3, x_1, x_2\}. \\
\end{align*}
\]

Consequently, \( T^* \) is a HIST of \( G \).

If \( w_1z \in E(T') \), let \( T^* := T_H \cup \{w_1x_1, uv_1, uv_3, w_2v_2, w_2x_2\} \), as depicted in Figure 2. \( w_2 = v_3 \) may occur.) As in the previous case, we can show that \( T^* \) is a HIST of \( G \).

![Figure 2: \( uw_1 \notin E(T') \) or \( uw_1 \in E(T') \cap E(G) \)]
Subcase 2. \( uw_1 \in E(T') - E(G) \).

In this case, \( T_1 := T_H - uw_1 \) has exactly two components. We construct a HIST of \( G \) from \( T_1 \) according to whether \( uz \in E(T') \) or \( w_1z \in E(T') \).

If \( uz \in E(T') \), let \( T^* = T_1 \cup \{ uv_3, uv_1, v_1w_1, v_1x_1, w_2v_2, w_2x_2 \} \), as depicted in Figure 3. It is routine to check that \( T^* \) is a spanning tree of \( G \) and the following equalities/inequalities hold.

\[
\begin{align*}
        d_{T^*}(u) &= d_{T'}(u) - |{uw_1, uz}| + |{uw_1, w_3}| = d_{T'}(u) \neq 2 \\
        d_{T^*}(w_1) &= d_{T'}(w_1) - |{uw_1}| + |{v_1w_1}| = d_{T'}(w_1) \neq 2 \\
        d_{T^*}(w_2) &= \begin{cases} 
            d_{T'}(w_2) + |{w_2v_2, w_2x_2}| = d_{T'}(w_2) + 2 \neq 2, & \text{if } w_2 \in V(H); \\
            |{w_2v_2, w_2x_2, w_3}| = 3, & \text{if } w_2 = v_3.
        \end{cases} \\
        d_{T^*}(x) &= d_{T'}(x) \neq 2 \text{ for all other vertices } x \in V(H), \text{ and} \\
        d_{T^*}(x) &\neq 2 \text{ for each vertex } x \in \{v_1, v_2, v_3, x_1, x_2\}.
\end{align*}
\]

So, \( T^* \) is a HIST of \( G \).

In the case \( w_1z \in E(T') \), if \( v_1v_3 \in E(G) \), let

\[
T^* = T_1 \cup \{ w_1x_1, w_1v_1, v_1u, v_1v_3, w_2v_2, w_2x_2 \},
\]
as depicted in Figure 3. As in the previous case, we can show that \( T^* \) is a HIST of \( G \). To complete the proof, we show that the vertex \( v_1 \) can be chosen such that \( v_1v_3 \in E(G) \). If \( v_1v_3 \notin E(G) \), then both \( v_1 \) and \( v_3 \) are degree 1 vertices in \( F - u \cong P_3 \). So, \( v_2 \) is the degree 2 vertex in \( F - u \). If \( w_2 \in V(H) \), we would pick \( x_2 \) as \( x_1 \) and \( v_2 \) as our \( v_1 \) in the very beginning. So, \( w_2 = v_3 \). In this case, we can simply swap \( v_2 \) and \( v_3 \)(also \( w_2 \)) to ensure that \( v_1v_3 \in E(G) \).

\[\square\]

Clearly, the combination of the above three Lemmas gives Theorem 1.

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uz ∈ E(T′)

Figure 3: uw₁ ∈ E(T′) − E(G)

References


