Nonempty Intersection of Longest Paths in Series-Parallel Graphs

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In 1966 Gallai asked whether all longest paths in a connected graph have nonempty intersection. This is not true in general and various counterexamples have been found. However, the answer to Gallai's question is positive for several well-known classes of graphs, as for instance connected outerplanar graphs, connected split graphs, and 2-trees. A graph is series-parallel if it does not contain $K_4$ as a minor. Series-parallel graphs are also known as partial 2-trees, which are arbitrary subgraphs of 2-trees. We present two independent proofs that every connected series-parallel graph has a vertex that is common to all of its longest paths. Since 2-trees are maximal series-parallel graphs, and outerplanar graphs are also series-parallel, our result captures these two classes in one proof and strengthens them to a larger class of graphs. We also describe how this vertex can be found in linear time.

1 Introduction

A path in a graph is a \textit{longest path} if there exists no other path in the same graph that is strictly longer. The study of intersections of longest paths has a long history and, in particular, the

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question of whether every connected graph has a vertex that is common to all of its longest paths was raised by Gallai in 1966 \cite{11}. For some years it was not clear whether the answer is positive or negative until, finally, Walther \cite{27} found a graph on 25 vertices that answers Gallai’s question negatively.

Today, the smallest known graph answering Gallai’s question negatively is a graph on 12 vertices, found by Walther and Voss \cite{28}, and independently by Zamfirescu \cite{30} (see Figure 1). To see that the depicted graph does not have a vertex common to all longest paths, one can identify the three leaves to obtain the Petersen graph. Since the Petersen graph is hypohamiltonian, meaning that it does not have a Hamiltonian cycle but every vertex-deleted subgraph is Hamiltonian, it follows first that the length of a longest path is at most 9 (and it is exactly 9) and second that the intersection of all longest paths is empty. Note that the length of a longest path in the depicted graph can be at most 10 since at most two of its three leaves can be contained in a longest path. But any path of length 10 in the depicted graph would correspond to a Hamiltonian cycle in the Petersen graph.

![Figure 1: The counterexample of Walther, Voss, and Zamfirescu.](image)

These are by far not the only counterexamples. In fact, there are infinitely many (even planar) ones since every hypotraceable graph, meaning a graph having no Hamiltonian path whose all vertex-deleted subgraphs have a Hamiltonian path, is obviously a counterexample. Thomassen proved in \cite{24} that there are infinitely many such graphs.

Since the answer to Gallai’s question is negative in general, it seems natural to restrict the problem to subsets of a fixed size of all longest paths. It is well-known \cite{20} that any two longest paths of a connected graph share a common vertex. However, considering the intersection of more than two longest paths gets more intriguing. It is still unknown whether any three longest paths of every connected graph share a common vertex. Zamfirescu asked this question several times \cite{26,31} and it was mentioned at the 15th British Combinatorial Conference \cite{8}. It is presented as a conjecture in \cite{13} and as an open problem in the list collected by West \cite{29}. Progress in this direction was made by de Rezende, Martin, Wakabayashi, and the second author \cite{10}, who proved that, if all nontrivial blocks of a connected graph are Hamiltonian, then any three longest paths of the graph share a vertex. Skupień \cite{23} showed that, for every \( p \geq 7 \), there exists a connected graph such that \( p \) longest paths have no common vertex and every \( p - 1 \) longest paths have a common vertex.

Even though it seems as if the property of having a vertex common to all longest paths is too strong, there are some classes of connected graphs for which this property holds. A simple example is the class of trees since in a tree all longest paths contain its center(s). Moreover, Klavžar and Petkovšek \cite{17} proved that the intersection of all longest paths of a (connected)
split graph is nonempty. Furthermore, they showed that, if every block of a connected graph $G$ is Hamilton-connected, almost Hamilton-connected, or a cycle, then there exists a vertex common to all its longest paths. The latter result implies immediately that the answer to Gallai’s question is positive for the class of (connected) cacti, where a graph is a cactus if and only if every block is either a simple cycle or a vertex or a single edge.

In 2004, Balister, Győri, Lehel, and Schelp showed a similar result in [3] for the class of circular-arc graphs. There was a gap in their proof though, closed recently by Joos [15]. In 2013, de Rezende et al. proved the following two theorems.

**Theorem 1 ([10])**
For every connected outerplanar graph, there exists a vertex common to all its longest paths.

**Theorem 2 ([10])**
For every 2-tree, there exists a vertex common to all its longest paths.

Theorem 1 is a strengthening of a theorem by Axenovich [2], which states that any three longest paths in a connected outerplanar graph share a vertex. A more comprehensive survey of the problem of intersections of longest paths and cycles in general can be found in [22].

In this paper we treat the general case of nonempty intersection of all longest paths and prove that the answer to Gallai’s question is positive for the class of connected series-parallel graphs, settling a question raised in [10]. Since not only all trees and cacti but also outerplanar graphs and 2-trees are series-parallel, our result gives a unified proof for Theorems 1 and 2 and generalizes them to a larger class of graphs. We present two proofs of our result. The first one, given in Section 3, heavily relies on the structure of a so called underlying 2-tree (a 2-tree that contains the series-parallel graph as a spanning subgraph), whereas the second one, presented in Section 5, is based on a graph decomposition method introduced by Tutte [25]. We present both proofs because they both seem intriguing for the study of series-parallel graphs. While the first one shows that one can prove surprisingly strong results for a series-parallel graph only using the structure of an underlying 2-tree, the second one provides insight into the structure of Tutte’s decomposition of a series-parallel graph. We strongly believe that both methods can be useful to tackle open problems on series-parallel graphs in the future.

The rest of the paper is organized as follows. In Section 2, we give essential definitions and prove some statements that will be useful in what follows. In Section 3, we present the first proof of the main theorem. In Section 4, we present a specific way to decompose series-parallel graphs using a decomposition introduced by Tutte. We use this decomposition in Section 5 in our second proof of the main theorem. In Section 6, we describe a linear-time algorithm that finds a vertex contained in all longest paths in connected series-parallel graphs. Finally, in Section 7, we state several open problems concerning the intersection of longest paths in specific classes of graphs.

## 2 Preliminaries and definitions

We start with a few basic definitions which we use in the subsequent part of our paper. All graphs in this paper are either simple graphs or have at least three edges sharing two end
It is well-known that a graph is a partial 2-tree if and only if it is edge-adjacent and if it can be turned into $K_2$ by a sequence of the following operations: replacement of a pair of parallel edges with a single edge that connects their common endpoints or replacement of a pair of edges incident to a vertex of degree 2 other than $s$ or $t$ with a single edge. A graph $G$ is 2-terminal series-parallel if there are vertices $s$ and $t$ in $G$ such that $G$ is series-parallel with terminals $s$ and $t$. A graph $G$ is series-parallel if each of its 2-connected components is a 2-terminal series-parallel graph. (See [7, Sec. 11.2].)

A 2-tree can be defined in the following way. A single edge is a 2-tree. If $T$ is not a single edge, then $T$ is a 2-tree if and only if there exists a vertex $v$ of degree 2 such that its neighbors are adjacent and $T − v$ is also a 2-tree. A graph is a partial 2-tree if it is a subgraph of a 2-tree. (See [7, Sec. 11.1].) We say a partial 2-tree is trivial if it consists of a single vertex or a single edge. Note that every edge in a nontrivial 2-tree is contained in a triangle.

It is well-known that a graph is a partial 2-tree if and only if it is $K_3$-minor-free. Partial 2-trees are exactly the series-parallel graphs, and are also known for being the graphs with tree width at most 2. (See [7, Sec. 11.1].)

Next we present some notation we use in our proofs.

The length of a path $P$, denoted by $|P|$, is the number of edges in $P$. Let $L(G)$ denote the length of a longest path in the graph $G$. Let $\mathcal{L}(G)$ denote the set of all longest paths in $G$, that is, $\mathcal{L}(G) = \{P \mid P$ is a path in $G$ and $|P| = L(G)\}$. If the graph $G$ is clear from the context, we simply write $L$ for $L(G)$ and $\mathcal{L}$ for $\mathcal{L}(G)$.

By the intersection $P \cap P'$ of two paths $P$ and $P'$, we mean the intersection of their vertex sets. If $v$ is a vertex of the path $P$, we write $v \in P$. If $P$ and $P'$ have a common endpoint $x$ but no other common vertex, then the union $P \cup P'$ is simply defined as the path obtained by concatenating the path $P$ and the path $P'$ at the vertex $x$.

A subpath of a path $P$ is called a tail of $P$ if it contains an endpoint of $P$. Given a vertex $x$ in $P$, the path $P$ can be split into two subpaths $P'$ and $P''$ such that $P' \cap P'' = \{x\}$; we call them tails of $P$ starting at $x$. If $|P'| \geq |P''|$, then $P'$ is called a longer tail of $P$ starting at $x$. Let $H$ be a subgraph of $G$. If $P \cap H = \{x\}$, then $P$ is called a pending path of $H$ with origin $x$. If $P'$ is a tail of $P$ starting at $x$ and $P'$ is also a pending path of $H$ with origin $x$, we call it a pending tail of $P$ on $H$ with origin $x$.

Given a second path $Q$ such that $Q \cap P \neq \emptyset$ and such that at least one endpoint $x$ of $P$ is not contained in $Q$, we define the bridge path $P \backslash x Q$ as the path starting at the endpoint $x$ going along $P$ until the first intersection with $Q$.

For some subgraph $H$, we define $P[H]$ to be the induced subgraph of $P$ in $H$, that is, the collection of (maximal) subpaths of $P$ that lie in $H$. Note that this might be more than one path.

Next, we borrow some definitions and results presented by Tutte [25]. Let $G$ be a graph and $H$ be a subgraph of $G$. A vertex of attachment of $H$ in $G$ is a vertex of $H$ that is incident to some
edge of \( G \) that is not an edge of \( H \). Let \( T \) be a 2-tree and let \( x \) and \( y \) be two adjacent vertices in \( T \). An \( \{x, y\}\)-bridge in \( T \) is either the edge connecting \( x \) and \( y \) or a minimal subgraph \( B \) of \( T \) containing other than \( x \) and \( y \) and whose vertices of attachment are contained in \( \{x, y\} \). We say the edge connecting \( x \) and \( y \) is a trivial \( \{x, y\}\)-bridge. For each common neighbor \( z \) of \( x \) and \( y \), let \( B_{x,y,z}(T) \) be the \( \{x, y\}\)-bridge in \( T \) containing \( z \). If \( z_1 \neq z_2 \) are two common neighbors of \( x \) and \( y \), then \( B_{x,y,z_1}(T) \neq B_{x,y,z_2}(T) \), as otherwise \( T \) would contain a \( K_4 \)-minor. Then, by Theorem I.51 [25], we have \( B_{x,y,z_1}(T) \cap B_{x,y,z_2}(T) \subseteq \{x, y\} \). The interior of the non-trivial bridge \( B_{x,y,z}(T) \) is the graph \( B_{x,y,z}'(T) = B_{x,y,z}(T) - \{x, y\} \).

In what follows, for a series-parallel graph \( G = (V, E) \), we let \( T(G) = (V, F) \) denote an arbitrary but fixed 2-tree that contains \( G \) as a spanning subgraph. We say an underlying edge/triangle of \( G \) is an edge/triangle in \( T(G) \) independent of its existence in \( G \).

We denote by \( C_{x,y,z}(G) \) the maximal subgraph of \( G \) contained in \( B_{x,y,z}(T(G)) \). Note that such a subgraph of \( G \) may be disconnected. We call this subgraph the component of \( G \) generated by the underlying edge \( \{x, y\} \) in direction \( z \). Similarly, \( C^0_{x,y,z}(G) = C_{x,y,z}(G) - x - y \) is called the interior of the component \( C_{x,y,z}(G) \). Again, if the graph is clear from the context, we write \( C_{x,y,z} = C_{x,y,z}(G) \).

Also, let \( \mathcal{C}_{x,y}(G) = \{C_{x,y,z}(G) : z \text{ is adjacent to } x \text{ and } y \text{ in } T(G)\} \) be the set of all components generated by the underlying edge \( \{x, y\} \). Further, define \( \mathcal{C}_{x,y}(G) = \mathcal{C}_{x,y}(G) \setminus \{C_{x,y,z}(G)\} \) for an underlying triangle \( \{x, y, z\} \).

A set of vertices \( W \) is called a Gallai set (for \( G \)) if \( W \cap P \neq \emptyset \) for all \( P \in \mathcal{L}(G) \). If \( W \) is a Gallai set and if its vertices are pairwise connected by underlying edges, then \( |W| \leq 3 \). In this case, we call this set an underlying Gallai edge and an underlying Gallai triangle when \( W \) has size two or three, respectively.

We say a vertex \( v \in V \) is a Gallai vertex if \( \{v\} \) is a Gallai set. Note that the intersection of all longest paths of a graph \( G \) is nonempty if and only if \( G \) has a Gallai vertex.

For an underlying edge \( \{u, v\} \), let \( \mathcal{L}_{uv} = \{P \in \mathcal{L} : u, v \in P\} \) and \( \mathcal{L}_{uv}^P = \{P \in \mathcal{L} : u \in P, v \notin P\} \).

Similarly, for an underlying triangle \( \triangle = \{u, v, w\} \), we define \( \mathcal{L}_{uvw} = \{P \in \mathcal{L} : u, v, w \in P\} \), \( \mathcal{L}_{uvw}^P = \{P \in \mathcal{L} : u, v \in P, w \notin P\} \), and \( \mathcal{L}_{uvw}^P = \{P \in \mathcal{L} : u \in P, v, w \notin P\} \). Moreover, let \( \mathcal{L}_{(uvw)} = \{P \in \mathcal{L}_{uvw} : v \text{ is between } u \text{ and } w \text{ in } P\} \).

A block of \( G \) is a maximal 2-connected graph of \( G \). For the purpose of this paper, we assume that \( K_2 \) is 2-connected and call it the trivial 2-connected graph. A graph \( B_1v_1B_2v_2\cdots v_{k-1}B_k \) which consists of an alternating sequence of blocks and cut vertices is called a block chain if each pair of adjacent blocks \( B_i \) and \( B_{i+1} \) share the distinct cut vertex \( v_i = V(B_i) \cap V(B_{i+1}) \) for \( i = 1, 2, \ldots, k-1 \).

If \( G \) is a 2-connected graph, a 2-cut of \( G \) is a 2-vertex set \( W \subseteq V(G) \) such that \( G - W \) is not connected. A 2-vertex set \( \{u, v\} \) is called a 2-separation if \( \{u, v\} \) has at least three bridges, which is equivalent to having one of the following three conditions held: (1) \( G - \{u, v\} \) has at least three components, (2) \( G - \{u, v\} \) has exactly two components and \( |E_G(u, v)| = 1 \), or (3) \( |E_G(u, v)| \geq 2 \) and \( |E(G)| \geq 3 \). The last condition indicates that a 2-separation \( \{u, v\} \) is not a 2-cut only if there are multiple edges between \( u \) and \( v \) and \( |E(G)| \geq 3 \).
Following Tutte [25], we call a 2-vertex graph with at least three edges between them a bond. A Θ-graph is a simple graph consisting of three internally disjoint paths sharing two end vertices.

We end this section with the following auxiliary results that will be useful in Section 3. Note that the first four results hold for general graphs, not only for series-parallel graphs.

**Proposition 3 ([20])**

Any two longest paths in a connected graph share a common vertex.

**Lemma 4**

In a graph, let \( P_1 \) and \( P_2 \) be two paths with tails \( R_1 \) and \( R_2 \), respectively (that is, subpaths containing an endpoint of \( P_1 \) or \( P_2 \)) such that \( R_1 \cap P_2 = \emptyset \) and \( R_2 \cap P_1 = \emptyset \). If there exists a connecting path \( P \) such that \( \emptyset \neq P \cap P_1 \subseteq R_1 \) and \( \emptyset \neq P \cap P_2 \subseteq R_2 \), then \( P_1 \) and \( P_2 \) cannot both be longest paths.

**Proof.** Assume for a contradiction that both \( P_1 \) and \( P_2 \) are longest paths. For \( i \in \{1, 2\} \), let \( R'_i \) denote the other tail of \( P_i \) so that \( P_i = R_i \cup R'_i \) and \( R_i \) and \( R'_i \) intersect at only one vertex. By assumption, both \( R_1 \) and \( R_2 \) intersect \( P \). Hence, there exist vertices \( x \) and \( y \) such that \( x \in R_1 \cap P \), \( y \in R_2 \cap P \), and the interior of the subpath of \( P \) starting at \( x \) and ending in \( y \) does not contain vertices in \( P_1 \) or \( P_2 \). Let \( Q_1 \) denote the path obtained from going along \( R'_1 \), along \( R_1 \) until \( x \), along \( P \) until \( y \), and then along \( R_2 \) until the endpoint that is not in \( R'_2 \). Let \( Q_2 \) denote the path obtained from going along \( R'_2 \), along \( R_2 \) until \( y \), along \( P \) until \( x \), and then along \( R_1 \) until the endpoint that is not in \( R'_1 \). Now, as \( |Q_1| + |Q_2| > |P_1| + |P_2| \), we have that \( |Q_1| > |P_1| \) or \( |Q_2| > |P_2| \), a contradiction. \( \square \)

![Figure 2: Situation for Lemmas 4 and 5.](image)

**Lemma 5**

In a graph, let \( P_1 \) and \( P_2 \) be two paths that share a common vertex \( z \) and let \( R_1 \) and \( R_2 \) be two subpaths of \( P_1 \) and \( P_2 \), respectively, both having \( z \) as an endpoint, such that \( R_1 \cap P_2 = \{z\} \) and \( R_2 \cap P_1 = \{z\} \). If there exists a connecting path \( P \) such that \( z \notin P \), \( \emptyset \neq P \cap P_1 \subseteq R_1 \), and \( \emptyset \neq P \cap P_2 \subseteq R_2 \), then \( P_1 \) and \( P_2 \) cannot both be longest paths.

**Proof.** Assume for a contradiction that both \( P_1 \) and \( P_2 \) are longest paths. By assumption, both \( R_1 \) and \( R_2 \) intersect \( P \) in a vertex other than \( z \). Hence, there exist vertices \( x \) and \( y \) distinct from \( z \) such that \( x \in R_1 \cap P \), \( y \in R_2 \cap P \), and the interior of the subpath of \( P \) starting at \( x \) and ending in \( y \) does not contain vertices in \( P_1 \) or \( P_2 \). Let \( \tilde{R}_1 \) denote the path starting at \( z \), going along \( R_1 \), and ending in \( x \), and let \( \tilde{R}_2 \) denote the path starting at \( z \), going along \( R_2 \), and ending in \( y \). Let \( R'_1 \) and \( R'_2 \) denote the tails of \( P_1 \) and \( P_2 \) starting at \( z \) not containing \( R_1 \) and \( R_2 \),
respectively. If $|\tilde{R}_1| \geq |\tilde{R}_2|$, then by combining $R'_2$, $\tilde{R}_1$, the subpath of $P$ starting at $x$ and ending in $y$, and the tail of $P_2$ starting at $y$ and not containing $z$, we get a path strictly longer than $P_2$, a contradiction. If, on the other hand, $|\tilde{R}_1| < |\tilde{R}_2|$, then by combining $R'_1$, $\tilde{R}_2$, the subpath of $P$ starting at $y$ and ending in $x$, and the tail of $P_1$ starting at $x$ and not containing $z$, we get a path strictly longer than $P_1$, a contradiction. □

Observe that Lemma 4 is not a consequence of Lemma 5. Indeed, in the situation of Lemma 4, starting from $x$ and going along $P_1$, the first common vertex with $P_2$ might not be the same as the first common vertex of $P_2$ with $P_1$, starting from $y$. Thus, the vertex $z$ required in Lemma 5 might not exist.

At some points of the proofs in the next section, we are in a situation where one of the two lemmas above apply. The next corollary describes this situation.

**Figure 3:** Situation for Corollary 6.

Corollary 6

In a graph, let $P_1$ and $P_2$ be two paths that share a common vertex $z$ and let $R_1$ be a tail of $P_1$ starting at $z$. Let $R_2$ be a union of pairwise internally vertex disjoint subpaths of $P_2$ (that is, they may have common endpoints) such that all paths in $R_2$ have as one endpoint $z$ or an endpoint of $P_2$. Suppose $R_1 \cap P_2 = \{z\}$ and $R_2 \cap P_1 \subseteq \{z\}$. If there exists a connecting path $P$ such that $z \notin P$, $\emptyset \neq P \cap P_1 \subseteq R_1$, and $\emptyset \neq P \cap P_2 \subseteq R_2$, then $P_1$ and $P_2$ cannot both be longest paths.

**Proof.** There exist vertices $x \in P \cap R_1$ and $y \in P \cap R_2$ such that the interior of the subpath $P^{x,y}$ of $P$ starting at $x$ and ending in $y$ does not contain any other vertices in $P_1$ or $P_2$. Let $R'_2$ be the path in $R_2$ that contains $y$. If $R'_2$ contains $z$ then the statement follows from Lemma 5 for longest paths $P_1$ and $P_2$ with their subpaths $R_1$ and $R'_2$, respectively, and connecting path $P^{x,y}$. Otherwise, the statement follows from Lemma 4 again for longest paths $P_1$ and $P_2$, tails $R_1$ and $R'_2$, and connecting path $P^{x,y}$. □

The next two results are specific for series-parallel graphs.

Lemma 7

Let $\Delta = \{v_1, v_2, v_3\}$ be an underlying triangle in a connected series-parallel graph $G$. If $R_i$ is a path in $G$ with $v_i$ as an endpoint and $R_i \cap \Delta = \{v_i\}$ for each $i \in \{1, 2, 3\}$, then only one of the sets $R_1 \cap R_2, R_1 \cap R_3$, and $R_2 \cap R_3$ can be nonempty. Furthermore, if $R_i \cap R_j \neq \emptyset$, then $R_i \cup R_j \subseteq C$ for some component $C \in \mathcal{G}_{[v_i, v_j]}$. v_i.

**Proof.** For the first statement, assume without loss of generality that $R_1 \cap R_2 \neq \emptyset$. Then we have a path $S$ from $v_1$ to $v_2$ consisting of $R_1 \setminus v_1 R_2$ and the tail of $R_2$ containing $v_2$. Note
that this path does not use \( v_3 \) or the underlying edge \( \{v_1, v_2\} \) because \( R_i \) contains only \( v_i \) in \( \Delta \) for \( i = 1, 2 \).

If additionally \( R_1 \cap R_3 \neq \emptyset \) or \( R_2 \cap R_3 \neq \emptyset \), then \( v_3 \) is connected to \( S \) by a path \( S' \cup S'' \), where \( S' \) is the shortest tail of \( R_3 \) from \( v_3 \) to a vertex \( u \) in \( R_1 \cup R_2 \), and \( S'' \) is a shortest path from \( u \) to \( S \) in the connected graph \( R_1 \cup R_2 \). Let \( x \) be the endpoint of \( S'' \) in \( S \). Observe that \( x \) is an internal vertex of \( S \). So \( \{x, v_1, v_2, v_3\} \) determines a \( K_4 \)-minor in \( T(G) \), a contradiction.

For the second statement, suppose that \( R_i \) and \( R_j \) intersect. Obviously, \( H = (R_i \cup R_j) - \{v_i, v_j\} \) must lie in the interior of a \( \{v_i, v_j\} \)-bridge \( B \) of \( T(G) \). Also, the edge \( \{v_i, v_j\} \) is a cut set in \( T(G) \), separating \( v_k \) from \( H \). Otherwise we would have three paths as above, namely \( R_i, R_j \), and the path from \( v_k \) to \( H \) avoiding \( v_i \) and \( v_j \), and at least two of the pairs within these three paths would intersect. Therefore \( B \in \mathcal{G}_{\{v_i, v_j\}v_k}(T(G)) \) and thus \( R_i \cup R_j \subseteq C = G[V(B)] \in \mathcal{G}_{\{v_i, v_j\}v_k}(G) \). \( \square \)

**Lemma 8**

Let \( \Delta = \{v_1, v_2, v_3\} \) be an underlying triangle in a connected series-parallel graph \( G \), and \( R \) be a path in \( G \) with \( v_i \) as endpoint and \( R \cap \Delta = \{v_i\} \), for some \( i \) in \( \{1, 2, 3\} \). Let \( j \) and \( k \) be such that \( \{i, j, k\} = \{1, 2, 3\} \). If \( S_1 \) is a path with endpoints \( v_i \) and \( v_j \) such that \( S_1 \cap \Delta = \{v_i, v_j\} \), and \( S_2 \) is a path with endpoints \( v_j \) and \( v_k \) such that \( S_2 \cap \Delta = \{v_j, v_k\} \), then \( R \cap S_2 = \emptyset \) and \( S_1 \cap S_2 = \{v_j\} \).

**Proof.** Assume for a contradiction that there is some vertex \( x \in R \cap S_2 \). Split the path \( S_2 \) at \( x \) and look at the two tails \( S_2^j \) and \( S_2^k \) starting at \( x \) and ending at \( v_j \) and \( v_k \), respectively. Now note that \( R, S_2^j, \) and \( S_2^k \) are three paths as in Lemma 7 and they all intersect at \( x \), a contradiction.

Similarly, if \( y \in (S_1 \cap S_2) \setminus \{v_j\} \), split \( S_2 \) analogously at \( y \) obtaining \( S_2^j \) and \( S_2^k \). Then \( S_1 - v_j, S_2^j, \) and \( S_2^k \) are three paths as in Lemma 7 and they all intersect at \( y \), again a contradiction. \( \square \)

**3 First proof**

As we have already mentioned in Section 1, de Rezende at el. [10] proved that the intersection of all longest paths of a 2-tree is nonempty. In this section, we extend this result proving that all connected subgraphs of 2-trees, that is, all connected series-parallel graphs, have also this property. We proceed in four steps. First, we prove in Lemma 9 that there exists an underlying Gallai triangle. Then, we show in Lemma 10 that actually one underlying edge of this triangle is an underlying Gallai edge and there exists a component generated by this underlying edge that satisfies certain properties. In Lemma 12 we prove that either one of the endpoints of this underlying edge is a Gallai vertex or we can find an adjacent underlying Gallai edge and a strictly smaller component satisfying the same properties. By iterating, we end up with a Gallai vertex since we only consider finite graphs.

**Lemma 9**

In every nontrivial connected series-parallel graph \( G \), there exists an underlying Gallai triangle.

**Proof.** Take any underlying triangle \( \Delta_0 \) of a nontrivial connected series-parallel graph \( G \). Note that, in every connected series-parallel graph, every underlying edge is either a cut set of \( G \) or is contained in exactly one underlying triangle. Assume there exists a longest path \( P_0 \) in \( G \)
Analogously, a path vertex \( u \) be split at \( u \) and a vertex containing no vertex of \( \Delta \). Let \( \Delta \subseteq \{ v \} \) be an underlying Gallai triangle in a nontrivial connected series-parallel graph \( G \). If \( \Delta = \{ x, y, z \} \), then, for some \( u, v, w \) such that \( \{ u, v, w \} = \Delta \), there exists a Gallai edge \( e \) adjacent to both endpoints of \( e \) in direction \( z \) and a vertex containing no vertex of \( \Delta \). By Proposition 3, all longest paths must intersect \( P_0 \) and so they have at least one vertex in \( C_{e_0, z_0} \). Note that \( \Delta_1 = e_0 \cup \{ z_0 \} \) is a triangle in \( T(G) \) and thus an underlying triangle in \( C_{e_0, z_0} \). Now either all longest paths contain a vertex of \( \Delta_1 \) and we are done, or there exist a longest path \( P_1 \), an underlying edge \( e_1 \subseteq \Delta_1 \), where \( e_1 \neq e_0 \) and \( e_1 \) is a cut set, and a vertex \( z_1 \notin \Delta_1 \) such that \( z_1 \) is adjacent to both endpoints of \( e_1 \) in \( T(G) \) and \( P_1 \subseteq C_{e_1, z_1} \). Note that \( C_{e_1, z_1} \subseteq C_{e_0, z_0} \), as \( P_1 \) must intersect \( P_0 \) in \( C_{e_0, z_0} \) again by Proposition 3. Iteratively, obtain \( \Delta_2 \) and \( C_{e_2, z_2} \) and eventually a strictly decreasing sequence of components \( \Delta = \Delta_0 \supseteq C_{e_1, z_1} \supseteq C_{e_2, z_2} \supseteq \cdots \supseteq C_{e_k, z_k} \). Since \( G \) is finite, this process ends with some triangle \( \Delta = \Delta_k \) such that all longest paths contain one vertex of \( \Delta \).

Next we prove that one of the underlying edges of a Gallai triangle is an underlying Gallai edge and there exists a component generated by this underlying edge that satisfies certain properties.

**Lemma 10**

For every connected series-parallel graph \( G = (V, E) \), there exists a Gallai vertex, or an underlying Gallai edge \( \{ u, v \} \) and a component \( C \in \mathcal{C}_G \) such that, for every pair \( (P, P') \in (\mathcal{L}_{u,v} \times \mathcal{L}_{v,w}) \cup (\mathcal{L}_{u,v} \times \mathcal{L}_{u,v}) \cup (\mathcal{L}_{v,w} \times \mathcal{L}_{v,w}) \), there exists a vertex in \( C \cap P \cap P' \).

Before presenting the proof of Lemma 10, we prove an intermediate result stated in the next lemma. Throughout the next proofs, we keep Lemmas 7 and 8 in mind and use them implicitly whenever we claim that certain constructions are indeed paths and whenever we claim that a path lies in a certain component.

For the proofs of Lemmas 11 and 12, we use the following notation. Every path \( P \in \mathcal{L}_{u,v} \) can be split at \( u \) and \( v \), resulting in three subpaths. Let \( P(u) \) and \( P(v) \) be the tails of \( P \) starting at vertex \( u \) and vertex \( v \), respectively. The remaining subpath, joining \( u \) and \( v \), is denoted by \( P(u,v) \). Analogously, a path \( P \in \mathcal{L}_{uvw} \) is split into \( P(u) \), \( P(u,v) \), \( P(v,w) \), and \( P(w) \) (see Figure 4).

![Figure 4: Splitting \( P \in \mathcal{L}_{uvw} \) at vertices \( u \) and \( v \), and \( P \in \mathcal{L}_{uvw} \) at vertices \( u, v, \) and \( w \).](image)

**Lemma 11**

Let \( \Delta \) be an underlying Gallai triangle in a nontrivial connected series-parallel graph \( G \). If \( \mathcal{L}_{xyz} \neq \emptyset \) for every \( x, y, z \) such that \( \Delta = \{ x, y, z \} \), then, for some \( u, v, w \) such that \( \{ u, v, w \} = \Delta \),
there is a component $C \in \mathcal{C}_{[u,w]}$, such that for every pair

$$(P, P') \in \left( \bigcup_{\{x,y,z\} = \Delta} \mathcal{L}_{x,y,z} \times \mathcal{L}_{x,y,z} \right) \cup (\mathcal{L}_{uvw} \times \mathcal{L}_{uvw}) \cup (\mathcal{L}_{vvw} \times \mathcal{L}_{vvw})$$

there exists a vertex in $C^\circ \cap P \cap P'$.

**Proof (Lemma 11)** Let $P \in \mathcal{L}_{uvw} \cup \mathcal{L}_{uwv} \cup \mathcal{L}_{vwu}$, where $\{u, v, w\} = \Delta$, and $x \in \Delta$ be such that $x$ is in $P$ and $P^{(x)}$ is as long as possible. Without loss of generality, we may assume $P \in \mathcal{L}_{uvw}$ and $x = u$. In what follows, we use $P_\Delta$ to refer to an arbitrary path in $\mathcal{L}_{x,y,z}$ where $\{x, y, z\} = \Delta$.

First note that $P^{(u)}$ intersects every $P^{(w)}$, otherwise $P^{(u)} \cup P^{(u,v)} \cup P^{(v,w)} \cup P^{(w)}$ is a path of length strictly greater than $L$ by the choice of $P$. Thus, $P^{(u)}$ must lie in a component $C \in \mathcal{C}_{[u,w]}$. We will prove that $C$ has the property stated in the lemma.

We start by proving that each path in $\mathcal{L}_{uvw}$ intersects every path in $\mathcal{L}_{vwu}$, that is, we show that each $P^{(u)}$ intersects every $P^{(w)}$. Observe that $|P^{(u)}| = |P^{(w)}|$, otherwise $P^{(u)} \cup P^{(u,v)} \cup P^{(v,w)}$ would be a path of length strictly greater than $L$. So the argument previously applied to $P$, now with $P_w$ instead, implies that each $P^{(u)}$ intersects every $P^{(w)}$.

Now we prove that each path in $\mathcal{L}_{uvw} \cup \mathcal{L}_{vwu}$ intersects every path in $\mathcal{L}_{uvw}$. First note that each $P^{(u)}$ intersects $Q^{(u,w)}$ in $C^\circ$ for every $Q \in \mathcal{L}_{uvw} \cup \mathcal{L}_{uwv} \cup \mathcal{L}_{vwu}$, otherwise $P^{(u)} \cup Q^{(u,w)} \cup P^{(w)}$ is a path of length strictly greater than $L$ by the choice of $P$. As both $\mathcal{L}_{uwv}$ and $\mathcal{L}_{vwu}$ are nonempty, there exist at least one such $Q$ and one such $P_u$. So $Q$ and $P^{(u)}$ must intersect in $C^\circ$, otherwise we derive a contradiction from Lemma 5 for subpaths $P^{(u)}$ and $Q^{(u,w)}$, $z = w$, and a connecting path contained in $P^{(u)}$. Second, we prove that each $P^{(u)}$ and $P^{(w)}$ intersect every $Q \in \mathcal{L}_{(uvw)}$ in $C^\circ$. Observe that $|Q^{(u)}| \leq |P^{(u)}|$, otherwise either $Q^{(u)} \cup P^{(u,v)} \cup P^{(v)}$ or $Q^{(u)} \cup P^{(u,w)} \cup P^{(w)}$ is a path of length strictly greater than $L$ by the choice of $P$. If $|Q^{(u)}| < |P^{(u)}| = |P^{(w)}|$, then $P^{(u)}$ intersects $Q^{(w)}$, otherwise $P^{(u)} \cup Q^{(u,v)} \cup Q^{(v,w)} \cup Q^{(w)}$ is a path of length strictly greater than $L$. So $P^{(w)}$ and $Q$ must intersect in $C^\circ$, otherwise we derive a contradiction from Corollary 6 for longest paths $P_u$ and $Q$ with $z = w$, tail $P^{(w)}$, subpaths $Q[C]$, and connecting path $P^{(w)}$. If $|Q^{(u)}| = |P^{(u)}|$, then $Q^{(u)}$ intersects $P^{(w)}$, otherwise $Q^{(u)} \cup Q^{(u,v)} \cup P^{(v,w)} \cup P^{(w)}$ is a path of length strictly greater than $L$ by the choice of $P$. So $P^{(u)}$ and $Q$ must intersect in $C^\circ$, otherwise we derive a contradiction from Corollary 6 for longest paths $P_w$ and $Q$ with $z = u$, tail $P^{(w)}$, subpaths $Q[C]$, and connecting path $P^{(w)}$. \hfill \Box

**Proof (Lemma 10)** If $G$ is trivial, there exists a Gallai vertex. Otherwise, let $\Delta = \{u, v, w\}$ be an underlying Gallai triangle, which exists by Lemma 9.

First, we show that at least one of the edges of $\Delta$ is an underlying Gallai edge. Assume for a contradiction that no edge of $\Delta$ is an underlying Gallai edge, which means that there are three longest paths $P_u \in \mathcal{L}_{u,v,w}$, $P_v \in \mathcal{L}_{v,u,w}$, and $P_w \in \mathcal{L}_{w,u,v}$. Thus, there exist three distinct components $C_{uv}$, $C_{uw}$, and $C_{vw}$ generated by the underlying edges of $\Delta$ such that, for every $x, y \in \Delta$, all intersection points of $P_x$ and $P_y$ lie in the component $C_{xy}$. Without loss of generality, let $|P_u[C_{uw}]| \geq L/2$. Then, by combining the paths $P_u[C_{uv}]$, $P_u[C_{uw}] \times u$, $P_w$, and a longer tail of $P_w$, we obtain a path of length strictly greater than $L$, a contradiction. So, there exists an underlying Gallai edge in $\Delta$. 

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If all edges of $\Delta$ are underlying Gallai edges, then $L_{uv\pi} = L_{v\pi\pi} = L_{w\pi\pi} = \emptyset$. If moreover at least one among $L_{uv\pi}$, $L_{uw\pi}$, $L_{vw\pi}$ is empty, then one of the vertices in $\Delta$ is a Gallai vertex. Otherwise, we are in the situation of Lemma 11 and the statement of the lemma follows immediately.

Without loss of generality, we may assume $\{u, v\}$ is an underlying Gallai edge. Hence, $L_{w\pi\pi} = \emptyset$. Let $P_u \in L_{v\pi\pi} \cup L_{w\pi\pi}$ and $x \in \{u, v\}$ be such that $P_u$ has a tail starting at $x$ that is as long as possible. Without loss of generality, we may assume $x = u$ and thus $P_u \in L_{w\pi\pi}$. Let $P_u$ be such a longer tail starting at $u$. (If both tails of $P_u$ starting at $u$ have same length, choose $P_u$ to be any one of them.) Let $P'_u$ be the other tail of $P_u$. As $L_{w\pi\pi}$ is nonempty, $\{v, w\}$ is not an underlying Gallai edge. If all longest paths contain $u$, then we are done since $u$ is a Gallai vertex. Otherwise, $L_{v\pi}$ is nonempty.

Each $P_v \in L_{v\pi}$ intersects $P'_u$ because otherwise, by combining $P'_u$, $P''_u \times_u P_v$, and a longer tail of $P_v$, we get a path of length strictly greater than $L$ by the choice of $P_u$. If $\{u, v\}$ is the only underlying Gallai edge in $\Delta$, then $L_{uv\pi} \neq \emptyset$ and $P'_u$ has to lie in a component $C \in \mathcal{C}_{\{u,v\}|w}$, so that it intersects every $P_v \in L_{v\pi\pi}$. Otherwise, $\{u, w\}$ is also an underlying Gallai edge and $P'_u$ lies in a component $C$ either in $\mathcal{C}_{\{u,v\}|w}$ or in $\mathcal{C}_{\{u,w\}|v}$. (Note that in this case $L_{uv\pi} \neq \emptyset$.) Without loss of generality, we assume $C \in \mathcal{C}_{\{u,v\}|w}$. We claim that $C$ has the desired properties.

First, we prove that each path in $L_{v\pi}$ intersects in $C^\circ$ every path in $L_{u\pi} \cup L_{uv}$. Let $P_v \in L_{v\pi}$. Suppose that there exists a path $Q \in L_{u\pi} \cup L_{uv}$ such that $P_v$ does not intersect $Q$ in $C^\circ$. Either $Q$ intersects $P'_u$ in $C^\circ$, or $P'_u$ and both tails of $Q$ starting at $u$ have length $L/2$ (see Figure 5). In the former case, since $P'_u$ intersects both $P_v$ and $Q$ in $C^\circ$, we can apply Lemma 4 if $Q \in L_{u\pi}$ (with paths $P_v$ and $Q$ and connecting path $P'_u$) or Corollary 6 if $Q \in L_{uv}$ (with paths $P_v$ and $Q$, $z = v$, a suitable tail of $P_v$ starting at $v$, subpaths $Q[C]$, and a connecting path contained in $P'_u$) deriving a contradiction. So, suppose now that $P'_u$ and both tails of $Q$ starting at $u$ have equal length. The path $P_v$ intersects both tails of $Q$ starting at $u$ because otherwise such a tail of $Q$, $P'_u \times_u P_v$, and a longer tail of $P_v$ would be a path of length strictly greater than $L$. Therefore, and since the combination of $P'_u$ with one tail of $Q$ starting at $u$ and the combination of $P'_u$ with the other tail of $Q$ starting at $u$ are both longest paths, we can apply Lemma 5 with a connecting path contained in $P_v$ and $z = u$, deriving again a contradiction. Hence, each path in $L_{v\pi}$ intersects in $C^\circ$ every path in $L_{u\pi} \cup L_{uv}$.

Next, we prove that each path in $L_{uv}$ intersects in $C^\circ$ every path in $L_{u\pi}$. If $L_{uv} \neq \emptyset$, let $P$ be a path in $L_{uv}$ and $Q_v$ in $L_{u\pi}$. Assume that $P$ does not intersect $Q_v$ in $C^\circ$. Let $P_v \in L_{v\pi}$ and note that such a longest path must exist. Since $P_v$ intersects $Q_v$ in $C^\circ$ and $P$ in $C^\circ$, we derive a contradiction longest paths $Q_v$ and $P$ with $z = u$, a suitable tail of $Q_v$ starting at $u$, subpaths $P[C]$, and a connecting path contained in $P_v[C]$. □

Lemma 12
In a nontrivial connected series-parallel graph $G$, let $e = \{u, v\}$ be an underlying Gallai edge and $C \in \mathcal{C}_e$ be a component such that all pairs of longest paths mutually intersect in at least one vertex of $C$ and all pairs of longest paths in $L_{u\pi} \times L_{v\pi}, L_{u\pi} \times L_{uv}$, and $L_{v\pi} \times L_{uv}$ mutually intersect in $C^\circ$ (as in Lemma 10). Let $w$ be the unique vertex in $C$ adjacent in $T(G)$ to both $u$ and $v$. Then $u$, $v$, or $w$ is a Gallai vertex, or there is an underlying edge $f$ incident to $u$ or $v$ and a component $C_1 \in \mathcal{C}_f, C_1 \subseteq C$, with the properties of Lemma 10.
Figure 5: Left: $P_v$ and $Q$ do not intersect in $C$ (apply either Lemma 4 or Corollary 6); Right: $P_v$ intersects both tails of $Q$ outside of $C$ (apply Lemma 5).

Proof. Let $\Delta = \{u, v, w\}$. Assume neither $u$ nor $v$ are Gallai vertices (otherwise there is nothing more to prove). Thus, both $L_{uv}$ and $L_{v\pi}$ are nonempty. All pairs of paths in $L_{uv} \times L_{v\pi}$ intersect in $C^\circ$ by the assumptions of the lemma. By Lemma 7, all paths in $L_{uv}$ or all paths in $L_{v\pi}$ must contain $w$ since $C \notin C_{[u,v]}$. Without loss of generality we may assume that all paths in $L_{uv}$ contain $w$. Therefore, $L_{uvw} = \emptyset$ and $\{v, w\}$ is an underlying Gallai edge.

We distinguish two cases. First, we consider the case in which there exists a path in $L_{v\pi}$ that does not contain $w$ (that is, a path in $L_{v\piw}$) and then the case in which all paths in $L_{v\pi}$ contain $w$ (that is, $L_{v\piw} = \emptyset$). In both cases we show that either there exists a Gallai vertex, or an underlying Gallai edge and a component strictly smaller than $C$ that fulfill the requirements of Lemma 12.

Figure 6: The scenario of the proof of Lemma 12: Case 1 (left) and Case 2 (right).

Case 1. The set $L_{v\piw}$ is nonempty.

For every path $P_u \in L_{uv} = L_{uvw}$, the tail $P_u[w]$ must intersect every path in $L_{v\piw}$ by assumption. Let $C_1 \subseteq C$, $C_1 \in C_{[v,w]} \setminus u$ be the unique component where they mutually intersect.
We claim that the underlying edge \{v, w\} together with the component \(C_1\) fulfills the requirements of Lemma 12.

If \(\mathcal{L}_{uv}\) \(\neq \emptyset\), let \(P_u \in \mathcal{L}_{uv}\) and \(P_v \in \mathcal{L}_{vw}\). Assume for a contradiction that there exists a path \(P_u \in \mathcal{L}_{uv}\) such that \(P\) does not intersect \(P_v\) in \(C_1^o\). Note that \(P_u\) and \(P_v\) must intersect in \(C^o\) by assumption and hence \(P_u\) and \(P(v)\) are disjoint. Since \(P_v\) intersects \(P_v\) in \(C_1^o\) and \(P\) in \(C_u\) (at least in vertex \(v\)), we can apply Lemma 4 (with paths \(P_u\) and \(P_v\), tails \(P_u^{(w)}\) and \(P(v)\), and a connecting path contained in \(P_v[C_1]\)) to derive a contradiction.

Next, we prove that every path in \(\mathcal{L}_{vw}\) \(\cup \mathcal{L}_{uv}\) intersects every path in \(\mathcal{L}_{uw}\) \(\cup \mathcal{L}_{uv}\) in \(C_1^o\).

Let \(P \in \mathcal{L}_{vw} \cup \mathcal{L}_{uv}\) and \(P_v \in \mathcal{L}_{uv}\). Note that \(P_v\) intersects \(P\) in \(C^o\) by assumption if \(P \in \mathcal{L}_{uw}\). Otherwise, \(P \in \mathcal{L}_{vw}\), and they also intersect in \(C^o\), or not both \(P\) and \(P_v\) could be longest paths by Lemma 5 for \(z = v\), a suitable tail of \(P_v[C_1]\), tail \(P(v)_u\), and connecting path \(P(v)_v\) for some \(P_u \in \mathcal{L}_{uv}\). Furthermore, \(P\) must intersect \(P_v\) in \(C_1^o\). Otherwise, \(P_u\) would have a tail starting at \(v\) completely in a component \(C' \in \mathcal{G}[v,w]\), \(C' \neq C_1\). Then \(|P_v[C']| < L/2\) since \(P_v[C']\) is disjoint from \(P_u\) and so \(P_v[C'] \cup P_v[C_1] \cup P_u\), and a longer tail of \(P_u\) would be a path of length strictly greater than \(L\). But now, by combining \(P_v[C_1]\) and a longer tail of \(P\) starting at \(v\), we get a path of length strictly greater than \(L\), a contradiction. For every \(P_u \in \mathcal{L}_{uv}\), by applying Corollary 6 (with paths \(P_u\) and \(P_v\), \(z = w\), tail \(P(v)_u\), subpaths \(P_v[C_1]\), and connecting path \(P_v[C_1]\)), we can deduce that \(P\) intersects \(P_u\) in \(C_1^o\). Every \(P_v \in \mathcal{L}_{uw}\) must intersect \(P\) in \(C_1^o\), otherwise we get a contradiction by applying Corollary 6 (with paths \(P\) and \(P_w\), \(z = v\), tail \(P_w[C_1]\), subpaths \(P_v[C_1]\), and connecting path \(P(v)_u\)).

Case 2. The set \(\mathcal{L}_{uv}\) is empty.

If the set \(\mathcal{L}_{uv}\) is empty, then all longest paths contain \(w\), therefore \(w\) is a Gallai vertex and the requirements of Lemma 12 are fulfilled. So, from now on, we assume that the set \(\mathcal{L}_{uv}\) is non-empty.

First, we prove that, for each \(P \in \mathcal{L}_{uv}\), either \(P(v)\) intersects \(P_u^{(w)}\) for every \(P_u \in \mathcal{L}_{uw} = \mathcal{L}_{uw}\), or \(P(u)\) intersects \(P(v)_u\) for every \(P_v \in \mathcal{L}_{uv} = \mathcal{L}_{uv}\). Assume for a contradiction that there exist \(P \in \mathcal{L}_{uv}\), \(P_u \in \mathcal{L}_{uw}\), and \(P_v \in \mathcal{L}_{uv}\) such that \(P(u)\) does not intersect \(P(v)_u\) and \(P(u)\) does not intersect \(P(v)_u\). By the assumptions of the lemma, \(P\) has to intersect both \(P_u\) and \(P_v\) in \(C^o\). Therefore, \(P\) intersects \(P_u\) in the interior of some component of \(\mathcal{G}[u,w]_{uv}\), and \(P_u\) in the interior of some component of \(\mathcal{G}[v,w]_{uv}\). (See Figure 7.) First, suppose that \(P(u)\) and \(P(v)\) do not intersect.

By combining \(P(v)_u\), \(P(v)\), and \(P(v)_v\), and a longer tail of \(P_u\), we get a path that cannot be of length strictly greater than \(L\), hence \(|P(v)| < L/2\). Combining \(P(u)\cup P(v)_u\cup P(v)_w\cup P(v)_w\) would be a path of length strictly greater than \(L\) if \(|P(v)| \leq L/2\). However, by combining \(P(u)_v\), \(P(v)_u\), \(P(v)_w\), \(P(v)_w\), and \(P(u)_v\), we get a path of length strictly greater than \(L\), a contradiction. If, on the other hand, \(P(u)\) and \(P(v)\) intersect, then \(P(u)\) and \(P(v)\) are disjoint except for the vertex \(u\). Note that \(|P(u)| = |P(v)_u\cup P(v)_w|\) since otherwise \(P\) and \(P_v\) would not be longest paths. The combination of \(P(u)\) and \(P(u)\) is therefore a longest path in \(\mathcal{L}_{uv}\), which is a contradiction. Therefore, \(P(u)\) intersects \(P(v)_u\) or \(P(v)\) intersects \(P(v)_u\).

We claim that if \(R(u)\), for some longest path \(R\) in \(\mathcal{L}_{uv}\), does not intersect \(P(v)_u\) for some path \(P_u \in \mathcal{L}_{uv}\), then for every longest path \(P \in \mathcal{L}_{uv}\), the tail \(P(v)_u\) intersects \(P(v)_w\) for every path \(P_u \in \mathcal{L}_{uv}\). Indeed, let \(P\) be a path in \(\mathcal{L}_{uv}\). Assume for a contradiction that there
exists a path $P_u \in \mathcal{L}_{u\gamma}$ such that $P_v$ does not intersect $P_w$. Note that by the latter paragraph $P_u^{(w)}$ must intersect $R^{(v)}$ and $P_v^{(w)}$ must intersect $P_{(u)}^{(w)}$. If $P^{(v)}$ lies in some component of $\mathcal{C}_{|v,w]|u}$, then by Lemma 7 $P^{(v)}$ cannot intersect $R^{(u)}$ and $R^{(u,v)} - v$, and $R^{(v)}$ cannot intersect $P^{(u,v)} - v$ since $R^{(v)}$ also lies in a component of $\mathcal{C}_{|v,w]|u}$. Hence, we have $|P^{(v)}| = |R^{(v)}|$ and $Q = R^{(u)} \cup R^{(u,v)} \cup P^{(v)}$ is a path in $\mathcal{L}_{u\gamma}$ such that $Q^{(u)}$ does not intersect $P^{(w)}$, and $Q^{(v)}$ does not intersect $P^{(w)}$, a contradiction. Analogously, if $R^{(u)}$ lies in a component of $\mathcal{C}_{|u,w]|v}$, the path $R^{(u)} \cup P^{(u,v)} \cup P^{(v)}$ yields a contradiction. Thus, we may assume that both $R^{(u)}$ and $P^{(v)}$ lie in a component of $\mathcal{C}_{|u,v]|w}$ and are therefore disjoint from $P^{(u)}$ except for $u$, and from $R^{(v)}$ except for $v$, respectively. Now, we get a contradiction from Lemma 4 for longest paths $R$ and $P$, tails $R^{(v)}$ and $P^{(u)}$, and a connecting path contained in $(P^{(w)} \times_w R) \cup (P^{(w)} \times_w P)$.

Without loss of generality, we may assume that, for each $P \in \mathcal{L}_{u\gamma}$, the tail $P^{(v)}$ intersects $P^{(w)}$ for every $P_u \in \mathcal{L}_{u\gamma}$. Let $C_1 \in \mathcal{C}_{|v,w]|u}$ be the unique component where they mutually intersect. Note that the underlying edge $\{v, w\}$ is indeed an underlying Gallai edge since the set $\mathcal{L}_{u\gamma}$ is empty. Let $P \in \mathcal{L}_{u\gamma}$ and $P_u \in \mathcal{L}_{u\gamma}$ be arbitrary but fixed. Note that $P_u$ intersects $P$ in $C_1^{u}$. Assume for a contradiction that there exists a longest path $P_v \in \mathcal{L}_{v\gamma} \cup \mathcal{L}_{v\delta}$ such that $P$ and $P_v$ do not intersect each other in $C_1^{u}$. Then, not both $P$ and $P_v$ can be longest paths by Corollary 6 for $z = v$, tail $P^{(v)}$, subpaths $P_u[C_1]$, and connecting path $P^{(w)}_u$, a contradiction. Therefore, the underlying edge $\{v, w\}$ together with the component $C_1$ fulfills the requirements of Lemma 12. □

**Theorem 13**

For every connected series-parallel graph $G$, there exists a vertex $v$ such that all longest paths in $G$ contain $v$.

**Proof.** This follows from Lemma 10 and by iteratively applying Lemma 12 since $G$ is finite. □

4 A decomposition of $K_4$-minor-free graphs

We now proceed to prove Theorem 13 in a different manner, using a graph decomposition method introduced by Tutte [25]. Similar to decomposing a connected graph into blocks and cut vertices, Tutte introduced a method to decompose 2-connected graphs into 3-blocks: bonds, cycles, and 3-connected graphs. We modify this method particularly for series-parallel graphs.
For this, if there exists a path $P = v_0v_1\cdots v_m$, then, assuming that $i \leq j$, let $P^{[v_i,v_j]} = v_i v_{i+1} \cdots v_j$, $P(v_i,v_j) = P^{[v_i,v_j]} - \{v_i, v_j\}$, $P^{[v_i,v_j]} = P^{[v_i,v_j]} - \{v_j\}$, and $P^{[v_i,v_j]} = P^{[v_i,v_j]} - \{v_i\}$. Also, if there is a cycle $C = v_0v_1\cdots v_{m-1}v_m$, where $v_0 = v_m$, let $\overline{C}^{[v_i,v_j]} = v_i v_{i+1} \cdots v_j$ and $\overline{C}^{[v_i,v_j]} = v_j v_{j+1} \cdots v_i$, with the indices taken modulo $m$. Similarly, define $\overline{C}^{[v_i,v_j]}$ and $\overline{C}^{(v_i,v_j)}$.

The following observations imply that there are no 3-connected blocks (except $K_3$) when applying the Tutte Decomposition Algorithm to a series-parallel graph.

**Lemma 14**
Let $G$ be a connected series-parallel graph. If a subgraph $H$ of $G$ is a $\Theta$-graph consisting of three internally vertex-disjoint paths $P^{[u,v]}_1$, $P^{[u,v]}_2$, and $P^{[u,v]}_3$, then $P^{[u,v]}_1$, $P^{[u,v]}_2$, and $P^{[u,v]}_3$ are in three different components of $G - \{u, v\}$ provided they are not empty. Consequently, $\{u, v\}$ is a 2-separation of $G$.

**Proof.** Suppose to the contrary, there is a path $Q^{[x,y]}$ connecting $x \in P^{[u,v]}_1$ and $y \in P^{[u,v]}_2$, such that $Q^{[x,y]} \cap P^{[u,v]}_3 = \emptyset$. Then by contracting $P^{[u,v]}_1$ to $x$, $P^{[u,v]}_2$ to $y$, $P^{[u,v]}_3$ to $u$, and $Q^{[x,y]}$ to $x$, we get a $K_4$, giving a contradiction. \(\square\)

**Lemma 15**
Let $G$ be a 2-connected series-parallel graph but not a bond, and let $e = uv$ be an arbitrary edge of $G$. If $\{u, v\}$ is not a 2-separation of $G$, then $G - e$ is no longer 2-connected. In particular, if $G$ has at least 4 vertices, $G$ is not 3-connected.

**Proof.** Suppose on the contrary that $G - e$ is still 2-connected. By Menger’s Theorem, there exist two internally vertex-disjoint $(u, v)$-paths $P$ and $Q$. Now, $P \cup Q \cup \{e\}$ is a $\Theta$-graph. By Lemma 14, the paths $P^{(u,v)}$ and $Q^{(u,v)}$ are in two different components of $G - \{u, v\}$, which in turn shows that $\{u, v\}$ is a 2-separation of $G$, a contradiction. \(\square\)

**Lemma 16**
Let $G$ be a 2-connected graph and $\{u, v\}$ be a 2-separation of $G$. There is no 2-separation $\{x, y\}$ such that $x$ and $y$ are in different $\{u, v\}$-bridges of $G$.

**Proof.** Since $\{u, v\}$ is a 2-separation of $G$, there are three internally vertex-disjoint $(u, v)$-paths in $G$. So, for any two distinct vertices $x$ and $y$ in different $\{u, v\}$-bridges, $G - \{x, y\}$ does not separate $u$ and $v$. As $G$ is 2-connected, each graph induced on the vertex set of a $\{u, v\}$-bridge together with the edge $\{u, v\}$ is 2-connected, so $\{x, y\}$ is not a 2-cut of $G$, which in turn shows that $\{x, y\}$ is not a 2-separation of $G$. \(\square\)

In the following, we apply the Tutte’s Decomposition Algorithm (TDA) (see Chapter 3 in [25]) specifically to series-parallel graphs. Given a 2-connected graph $G$, we apply TDA to produce a set $\mathcal{D}(G)$ of 3-blocks, a set $\varphi(G)$ of virtual edges, and a rooted tree $T_3(G)$ with vertex set $\mathcal{D}(G)$. This algorithm is applied on an ordered pair $(G, e)$ where $e$ is an arbitrary edge of $G$, but both $\mathcal{D}(G)$ and $\varphi(G)$ are independent from the selection of edge $e$ as Tutte [25] pointed out.

**Tutte’s Decomposition Algorithm (TDA)**

Let $G$ be a nontrivial 2-connected series-parallel simple graph, and let $e \in E(G)$ be an arbitrary edge of $G$ with endpoints $u$ and $v$. Perform the following operations.
Lemma 17 is clearly true if $G$ is a cycle. We assume that $G$ is not a cycle and verify Lemma 17 according to the bonds produced in O-A or O-B.
In O-A, by Lemma 16, the bonds are $B_{uv}$, and the ones resulted in $G_i$ for $i = 1, 2, \ldots, k$. Note that \{u, v\} is a 2-separation of $G$. Since $u$ and $v$ are adjacent in $G_i$ and \{u, v\} is not a 2-separation of $G_i$, a pair of vertices in $G_i$ is a 2-separation of $G_i$ if and only if it is a 2-separation in $G$. Inductively, we can show that Lemma 17 holds.

In O-B, by Lemma 16, the bonds are $B_{v_i, v_{i+1}}$, and the ones resulted in $G_{i,j}$ for each $G_{i} \neq e_i$ with $1 \leq j \leq k_i$. Since each $G_i$ is 2-connected, there are two internally vertex-disjoint $(v_i, v_{i+1})$-paths $P_1$ and $P_2$ in $G_i$. Since these two paths and a $(v_i, v_{i+1})$-path along the other direction of $G_{uv}$ form a $\Theta$-graph, $P_1(v_i, v_{i+1})$ and $P_2(v_i, v_{i+1})$ are in different bridges of $\{v_i, v_{i+1}\}$ in $G_i$, which in turn shows that $\{v_i, v_{i+1}\}$ is a 2-separation of $G$. Since $G_{i,j}$ is 2-connected and $v_i$ and $v_{i+1}$ are adjacent in $G_{i,j}$ for each $1 \leq j \leq k_i$, a pair of vertices in $G_{i,j}$ form a 2-separation in $G_{i,j}$ if and only if they form a 2-separation in $G$. Inductively, we can show that Lemma 17 holds in this case.

Applying Lemma 17, we can show that $\mathcal{D}(G)$, $\varphi(G)$, and $T_3(G)$ are independent from the choice of the edge $uv$ and are uniquely determined by $G$. In fact, Tutte [25] showed that this statement is true for every 2-connected simple graph without the condition of being $K_4$-minor-free.

**Lemma 18**

For each vertex $v \in V(G)$, let $\mathcal{D}_v(G)$ denote the set of 3-blocks containing $v$ and $T_3[\mathcal{D}_v(G)]$ be the subgraph of $T_3(G)$ induced by $\mathcal{D}_v(G)$. Then $T_3[\mathcal{D}_v(G)]$ is a subtree of $T_3(G)$.

**Proof.** Since $T_3(G)$ is a tree, we only need to show that the subgraph induced by $\mathcal{D}_v(G)$ is connected. Otherwise, assume that there exist two 3-blocks $A, B \in \mathcal{D}_v(G)$ such that not all of the internal vertices of the $(A, B)$-path $D_1(= A)D_2 \cdots D_{m-1}D_m(= B)$ in $T_3(G)$ are in $\mathcal{D}_v(G)$. Let $i > 1$ be such that $v \in V(D_{i-1})$ and $v \notin V(D_i)$. Note that $T_3(G)$ is a bipartite graph with one class containing all bonds and the other containing all cycles, so this path is an alternating path of bonds and cycles. Recall that $V(B) \subset V(C)$ if bond $B$ and cycle $C$ are adjacent in $T_3(G)$. So $D_{i-1}$ is a cycle and $D_i$ is a bond. By Lemma 17, $V(D_i)$ is a 2-separation of $G$. As $v \notin V(D_i)$ and $v \in V(D_m)$, we have that $i < m$. Say $V(D_i) = \{u, w\}$. Since $T_3(G)$ is a tree, $D_1, D_2, \ldots, D_{i-1}$ and $D_{i+1}, D_{i+2}, \ldots, D_m$ are in two different components of $T_3(G) - D_i$. So $\bigcup_{j=1}^{i-1} V(D_j)$ and $\bigcup_{j=i+1}^{m} V(D_j)$ are in two different components of $G - \{u, w\}$, which contradicts that both contain the vertex $v$.

We now expand our consideration from 2-connected graphs to connected graphs. Let $G$ be a connected series-parallel graph. We obtain a Decomposition Tree, denoted as $T_G$, by the following steps.

**Step 1.** Decompose $G$ into blocks and cut vertices (see [25]) and obtain a block-cut vertex tree, say $T_2(G)$, by the following description.

The tree $T_2(G)$ is a bipartite graph with two partitions $(U, V)$ such that, for vertices in $U$, there is a 1-1 correspondence to the blocks of $G$ and such that, for vertices in $V$, there is a 1-1 correspondence to the cut vertices of $G$. For any $u \in U$ and $v \in V$, $uv \in E(T_2(G))$ if and only if the block corresponding to $u$ contains the cut vertex corresponding to $v$ in $G$. It is easy to see that $T_2(G)$ is a tree. Note that each block
with at least 4 vertices is a 2-connected series-parallel graph but not 3-connected by Lemma 15.

To simplify the notation, for each element \( X \in U \), we use \( X \) either as a vertex of \( T_2(G) \) or as a 2-connected subgraph of \( G \), with the meaning being clear from the context.

**Step 2.** For each \( X \in U \), we apply TDA to \( X \) and obtain \( T_3(X) \), the rooted tree of \( X \), with vertex set \( \mathcal{D}(X) \) and virtual edge set \( \varphi(X) \). Notice that if \( X = K_2 \), then \( T_3(X) = K_1 \), which is a single-vertex graph, and \( \mathcal{D}(X) \) consists of only cycles and bonds by the assumption that \( G \) is \( K_4 \)-minor-free.

For each \( X \in U \) and a cut vertex \( v \in X \), recall \( \mathcal{D}_v(X) = \{ B \in \mathcal{D}(X) \mid v \in B \} \), which we will call \( v \)-blocks in \( X \). By Lemma 18, \( T_3[\mathcal{D}_v(X)] \) is a subtree of \( T_3(X) \).

Let

\[
\mathcal{D}(G) = \bigcup_{X \in U} \mathcal{D}(X),
\]

which is a set consisting of \( K_2 \), cycles, and bonds obtained from Step 2. Recall that \( V \) is the set of cut vertices of \( G \) mentioned in Step 1. We modify \( T_2(G) \) into a graph \( T_G \) with vertex set \( V \cup \mathcal{D}(G) \) through the step below:

**Step 3.** For each \( X \in U \subset V(T_2(G)) \), replace \( X \) by the Tutte decomposition rooted tree \( T_3(X) \) of it; then, for each \( v \in V \), if \( vX \in E(T_2(G)) \), \( X \) is a block containing \( v \). In this case, we let \( v \) be adjacent to a vertex in \( T_3[\mathcal{D}_v(X)] \). (Note that to which specific vertex in \( T_3[\mathcal{D}_v(X)] \) we join \( v \) is not essential to our proof.) Denote the resulted tree by \( T_G \) and call it a decomposition tree of \( G \).

Given a subgraph \( H \) of \( G \), let \( V_{T_G}(H) = \{ X \in V(T_G) \mid X \cap V(H) \neq \emptyset \} \) and \( T_H = T_G[V_{T_G}(H)] \).

**Lemma 19**

The following two statements hold.

1. For each vertex \( v \in V(G) \), \( T_v \) is a subtree of \( T_G \);
2. For every edge \( e = uv \in E(G) \), \( T_u \cap T_v \neq \emptyset \).

**Proof.** (1) If \( v \) is not a cut vertex of \( G \), then \( v \) is contained in exactly one block of \( G \). We thus know \( T_v \) is connected by Lemma 18. Assume \( v \) is a cut vertex. Hence, for each block \( X \) of \( G \) which contains \( v \), \( T_G[\mathcal{D}_v(X)] \) is a subtree. Then \( T_v \) is the graph obtained by taking the union of all \( T_G[\mathcal{D}_v(X)] \) and adding \( v \) and edges joining \( v \) to a vertex of \( T_G[\mathcal{D}_v(X)] \), for each of the subtrees \( T_G[\mathcal{D}_v(X)] \). It is easy to see that \( T_v \) is a connected graph.

(2) For each edge \( e \in E(G) \), there exists exactly one block of \( G \), say \( B \), containing \( e \). If \( B = e \), then \( e \in T_u \cap T_v \). Otherwise, there are at least 3 vertices in \( B \). If \( \{u, v\} \) is not a 2-separation of \( G \), there is a unique cycle in \( \mathcal{D}(B) \) containing \( e \); otherwise, there is a bond in \( \mathcal{D}(B) \) containing \( e \). In either case, we have \( T_u \cap T_v \neq \emptyset \). \( \square \)
5 Second proof

In this section we present a second proof of Theorem 13. It follows from a sequence of claims. Let \( G \) be a connected simple series-parallel graph and \( \mathcal{L} \) be the set of all longest paths in \( G \).

Claim 20
We may assume that there exists a cycle \( C \in \mathcal{D}(G) \) such that \( P \cap C \neq \emptyset \) for each \( P \in \mathcal{L} \).

Proof. Let \( T_G \) be a decomposition tree of \( G \). For each longest path \( P \), recall \( V_{T_G}(P) = \{ X \in V(T_G) | X \cap V(P) \neq \emptyset \} \) and \( T_P = T_G[V_{T_G}(P)] \). By Lemma 19, \( T_P \) is connected and thus a subtree of \( T_G \). For any two longest paths \( P \) and \( Q \) in \( \mathcal{L} \), we have \( T_P \cap T_Q \neq \emptyset \) since \( P \cap Q \neq \emptyset \). Let \( T_{\mathcal{L}} = \{ T_P : P \in \mathcal{L} \} \). It is well-known that a family of subtrees of a tree has the Helly property (see problem 18 on p. 49 of [18]), so there is a vertex \( B \in V(T_G) \) such that \( B \in \bigcap_{P \in \mathcal{L}} T_P \). By the construction of \( T_G \), there are four possibilities for \( B \): a cut vertex of \( G \), a block \( K_2 \) of \( G \), a bond, or a cycle from \( \mathcal{D}(G) \). We may assume that \( B \) is a cut vertex, a cut edge, or a bond.

If \( B \) is a cut vertex of \( G \), then \( B \subset \bigcap_{P \in \mathcal{L}} P \), so Theorem 13 holds.

If \( B = \{x,y\} \) is a cut edge of \( G \), we may assume, without loss of generality, \( x \in \bigcap_{P \in \mathcal{L}} P \), so Theorem 13 holds. Indeed, we assume this, for otherwise there exist two longest paths \( P, Q \in \mathcal{L} \) such that \( x \in P \) but \( y \notin P \) and \( x \notin Q \) with \( y \in Q \). Since \( P \cap Q \neq \emptyset \), there is a vertex \( z \in P \cap Q \). Then \( \{x,y,z\} \) contains a triangle-minor, which contradicts \( \{x,y\} \) being a cut edge in \( G \).

Suppose \( B \) is a bond. Since \( G \) is a simple graph, let \( C \) be a cycle adjacent to \( B \) in \( T_G \). We have \( V(B) \subseteq V(C) \), which in turn gives Claim 20. \( \square \)

In what follows, the notation \( C \) is reserved for a cycle \( C \in \mathcal{D}(G) \) such that every longest path contains a vertex of \( C \). We want to show that \( C \) contains a Gallai vertex.

Let \( uv \in E(C) \). If \( \{u,v\} \) is not a 2-cut in \( G \), let \( G_{uv} \) be just the edge \( uv \). Otherwise, if \( \{u,v\} \) is a 2-cut in \( G \), then \( uv \in E(C) \) is a virtual edge while the possible real edge between \( u \) and \( v \) is in the corresponding bond. Following TDA and by Lemma 17, \( \{u,v\} \) is a 2-separation of \( G \). In this case, let \( G_{uv} \) be the subgraph of \( G \) obtained by deleting all components of \( G - \{u,v\} \) containing a vertex of \( C - \{u,v\} \). Since \( \{u,v\} \) is a 2-separation, there are two \( (u,v) \)-paths \( R^{(u,v)} \) and \( S^{(u,v)} \) in \( G_{uv} \). By Lemma 14, paths \( R^{(u,v)} \) and \( S^{(u,v)} \) are in different components of \( G - \{u,v\} \). We call \( R^{(u,v)} \) and \( S^{(u,v)} \) connectors of \( G_{uv} \).

The following two claims follow directly from TDA.

Claim 21
For any two distinct edges \( uv, pq \in E(C) \), we have \( V(G_{uv}) \cap V(G_{pq}) \subseteq \{u,v\} \cap \{p,q\} \) and \( E(G_{uv} - \{u,v\}, G_{pq} - \{p,q\}) = \emptyset \) (see Figure 8).

Claim 22
If \( p^{(u,v)} \) is a path in \( G \) with \( p^{(u,v)} \cap C = \{u,v\} \), then \( u \) and \( v \) are two consecutive vertices on \( C \).

Claim 23
If \( P \in \mathcal{L} \) is a longest path in \( G \) such that \( P \) has at least one end, say \( u \), on \( C \), then both its predecessor \( u^- \) and successor \( u^+ \) along \( C \) are also on \( P \).
Proof. Suppose, on the contrary, that \( u^+ \notin P \). If \( \{u, u^+\} \in E(G) \), then \( P \cup \{uu^+\} \) is a strictly longer path, a contradiction. Otherwise, consider \( R^{(u,u^+)} \) and \( S^{(u,u^+)} \). Since \( R^{(u,u^+)} \) and \( S^{(u,u^+)} \) are in different components of \( G - \{u, v\} \), one of them, say \( R^{(u,u^+)} \) is vertex-disjoint from \( P \). Thus \( P \cup R^{(u,u^+)} \) is a strictly longer path than \( P \), a contradiction. \( \square \)

For any two vertices \( x, y \in C \), we use \( \overrightarrow{C}_{[x,y]} \) (resp. \( \overleftarrow{C}_{[x,y]} \)) to denote a path obtained from \( \overrightarrow{C}^{[x,y]} \) (resp. \( \overleftarrow{C}^{[x,y]} \)) by replacing each edge \( \{u, v\} \in E(C) \) by \( R^{(u,v)} \) or \( S^{(u,v)} \) whenever \( \{u, v\} \) is a 2-separation (see Figure 9.)

Let \( \mathcal{L}_1 = \{P|P \in \mathcal{L} \text{ and } P \text{ has at least one end vertex on } C\}. \)

Claim 24
For every \( P \in \mathcal{L}_1 \), \( V(C) \subset V(P) \), so \( V(C) \subset \bigcap_{P \in \mathcal{L}_1} V(P) \).

Proof. Let \( P \in \mathcal{L}_1 \) with endpoint \( u \) on \( C \), and set \( Q = P - \{u\} \). By Claim 23, path \( Q \) contains both \( u^+ \) and \( u^- \), but not \( u \). By Claim 22, traveling along \( Q \) from \( u^+ \) to \( u^- \) one must go through all vertices in \( V(C) - u \), so \( V(C) \subset V(P) \). \( \square \)

Following Claim 24, we may assume \( \mathcal{L}_2 := \mathcal{L} \setminus \mathcal{L}_1 \neq \emptyset \). If \( |\mathcal{L}_2| \leq 1 \), then Theorem 13 holds. So, we may assume \( |\mathcal{L}_2| \geq 2 \). Moreover, we have \( \bigcap_{P \in \mathcal{L}_2} P \cap C = \bigcap_{P \in \mathcal{L}_2} P \cap C \). Notice that each path in \( \mathcal{L}_2 \) has exactly two pending paths of \( C \).
Claim 25
If \(|P \cap C| \geq 2\) for every \(P \in \mathcal{L}_2\), then \(\bigcap_{P \in \mathcal{L}} P \neq \emptyset\).

Proof. Let \(L\) be a longest pending path of \(C\), and let \(z\) be the origin (the common vertex of \(L\) and \(C\)) of \(L\). We claim \(z \in \bigcap_{P \in \mathcal{L}} P\). Suppose this is not true. By Claim 24, there exists \(P \in \mathcal{L}_2\) such that \(z \notin P\). Let \(P'\) and \(P''\) be the two pending tails of \(P\) on \(C\) with origins \(u_1\) and \(u_2\), respectively. If \(P' \cap L \neq \emptyset\), then \([u_1, z] \in E(C)\) and both \(P'\) and \(L\) are subgraphs of \(G_{u_1 z}\) by Claim 21, which in turn shows that \(P'' \cap L = \emptyset\). Similarly, if \(P'' \cap L \neq \emptyset\), then \(P' \cap L = \emptyset\). We assume, without loss of generality, \(P'' \cap L = \emptyset\). We also assume that the segment \(\overline{C}^{[u_1,u_2]}\) does not contain the vertex \(z\). Then, \([u_1,u_2] \cap L = \emptyset\). Note that the paths \(P'' \cup P[u_2,u_1]\), \(\overline{C}^{[u_1,z]}\), and \(L\) are internally vertex-disjoint. Concatenating these paths together, we obtain a strictly longer path, which gives a contradiction.

Hence, we assume there exists \(P \in \mathcal{L}_2\) sharing exactly one vertex with \(C\). We let \(P \cap C = \{v\}\), and let \(P'\) and \(P''\) be the two tails of \(P\) split at \(v\) ending in \(v_1\) and \(v_2\), respectively.

Claim 26
\(v \in \bigcap_{Q \in \mathcal{L}} Q\).

Proof. Suppose to the contrary that there exists \(Q \in \mathcal{L}\) such that \(v \notin Q\). By Claim 24, neither of the two ends of \(Q\) is on \(C\). Thus \(Q\) has two pending tails, denoted as \(Q' = Q[w_1,u_1]\) and \(Q'' = Q[w_2,u_2]\), of \(C\) with origins \(w_1\) and \(w_2\), respectively. We assume, without loss of generality, the segment \(\overline{C}^{[w_1,w_2]}\) does not contain \(v\). Note that \(w_1 = w_2\) if and only if \(Q\) and \(C\) share exactly one vertex.

We will distinguish a few cases according to which one of the four pending tails \(P', P'', Q', \) and \(Q''\) is the longest. By the symmetry of \(P'\) and \(P''\) and the symmetry of \(Q'\) and \(Q''\), we only need to consider two cases according to whether \(P'\) or \(Q'\) is the longest one among the four pending tails. In fact, since \(w_1 = w_2\) may occur, the case that \(Q'\) is the longest one is more general than the case in which \(P'\) is the longest one. So we assume that \(Q'\) is the longest pending tail among \(P', P'', Q', \) and \(Q''\).

We claim that \(P' \cap Q' \neq \emptyset\) and \(P'' \cap Q' \neq \emptyset\). Indeed, assume for a contradiction that \(Q'\) and, without loss of generality, \(P'\) are disjoint. Also, assume that, without loss of generality, \(\overline{C}^{[w,v]} = x_0(= v)x_1 \cdots x_m(= w_1)\) contains at least three vertices. We consider the cases that both, only one, or none of the edges \(\{x_0, x_1\}\) and \(\{x_{m-1}, x_m\}\) are 2-cuts. First, assume that both edges are 2-cuts. Since the two \((x_0, x_1)\)-paths \(R(x_0, x_1)\) and \(S(x_0, x_1)\) are in two different \(\{x_0, x_1\}\)-bridges, we may assume \(R(x_0, x_1)\) and \(P'\) only share a common vertex \(x_0\). Similarly, we may assume that \(R(x_{m-1}, x_m)\) and \(Q'\) only share a common vertex \(x_m\). Then, \(P', R(x_0, x_1), \overline{C}^{[x_0,x_{m-1}]}, R(x_{m-1}, x_m)\), and \(Q'\) are internally vertex-disjoint. By concatenating these paths, we get a path strictly longer than \(P\), a contradiction. Without loss of generality assume that \(\{x_0, x_1\}\) is a 2-cut but \(\{x_{m-1}, x_m\}\) is not. Then the two \((x_0, x_1)\)-paths \(R(x_0, x_1)\) and \(S(x_0, x_1)\) are in two different \(\{x_0, x_1\}\)-bridges. This means that, without loss of generality, \(R(x_0, x_1)\) and \(P'\) only share a common vertex \(x_0\). Hence, \(P' \cup R(x_0, x_1) \cup \overline{C}^{[x_0,x_{m-1}]} \cup \{x_{m-1}, x_m\} \cup Q'\) is strictly longer than \(P\), again a contradiction. If both \(\{x_0, x_1\}\) and \(\{x_{m-1}, x_m\}\) are not 2-cuts, then \(P'\{x_0, x_1\} \overline{C}^{[x_0,x_{m-1}]} \{x_{m-1}, x_m\} Q'\) is strictly longer than \(P\), a contradiction. Hence, \(P' \cap Q' \neq \emptyset\) and \(P'' \cap Q' \neq \emptyset\). Along with TDA, this indicates that \(\{w_1, v\} \in E(C)\).

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Let \( x \) (respectively \( y \)) be the first vertex along \( P^{[v,v_1]} \) (respectively \( P^{[w,v_2]} \)) intersecting \( Q' \), that is, \( Q' \cap P^{[v,x]} = \{ x \} \) and \( Q' \cap P^{[w,y]} = \{ y \} \). Moreover, we assume without loss of generality that \( x \) is between \( w_1 \) and \( y \) on \( Q' \).

Since \( Q^{[w_1,x]} \cup P^{[x,v]} \) and \( Q^{[w_1,y]} \cup P^{[y,v]} \) are two \( (w_1,v) \)-paths in \( G_{w_1,y} \), and the three pending tails \( Q', P', \) and \( P'' \) are in the same \( \{ v,w_1 \} \)-bridge in \( G \), note that, if \( Q' \cap P' \neq \emptyset \), then \( w_2 = w_1 \).

Let \( \Theta_{vx} \) be the union of the paths \( P^{[v,x]} \), \( P^{[v,y]} \cup Q^{[y,x]} \), and \( Q^{[x,w_1]} \cup Q^{[w_1,w_2]} \cup C^{[w_2,v]} \). Clearly, \( \Theta_{vx} \) is a \( \Theta \)-graph. Applying Lemma 14 to \( \Theta_{vx} \), we get \( P^{[v,v_2]} \cap Q^{[w_1,x]} = \emptyset \), \( Q' \cap P'' = \emptyset \), and \( Q'' \cap P^{[v,x]} = \emptyset \).

We claim \( Q^{[x,u_1]} \cap P^{[x,v_1]} \neq \emptyset \). Otherwise, let \( R_1 = P^{[x,v_1]} \cup Q^{[x,u_1]} \) and \( R_2 = P^{[w_2,v]} \cup P^{[v,x]} \cup Q^{[x,w_1]} \cup Q^{[w_1,w_2]} \cup Q^{[w_2,u_2]} \) be two walks. Since we assume \( Q^{[x,u_1]} \cap P^{[x,v_1]} = \emptyset \), \( R_1 \) is a path. Note that \( P^{[v_2,v]} \cup P^{[v,x]} = P^{[v_2,x]} \) and \( Q^{[x,w_1]} \cup Q^{[w_1,w_2]} \cup Q^{[w_2,u_2]} = Q^{[x,u_2]} \). Applying Lemma 14 to \( \Theta_{vx} \), we have that \( P^{[v_2,x]} \) and \( Q^{[x,u_2]} \) are internally vertex-disjoint, so \( R_2 \) is also a path.

Thus, we have

\[
|R_1| + |R_2| = |P| + |Q|.
\]
By Claim 20 and the fact that \( R_1 \cap C = \emptyset \), we know \( R_1 \) is not a longest path, so \( |R_2| > |P| \), which also gives a contradiction.

Since \( Q'^{[x,u_1]} \cap P^{[x,v_1]} \neq \emptyset \), applying Lemma 14 to \( \Theta_v \) again, we get \( P^{[x,v_1]} \cap Q'^{[w_1,x]} = \emptyset \). Since \( R^{[v,w_1]} \) and \( S^{[v,w_1]} \) belong to two different \( \{v,w_1\}\)-bridges, we may assume that \( R^{[v,w_1]} \) is not in the \( \{v,w_1\}\)-bridge containing \( x \). Let \( \overline{C}^{[w_1,v]} \) be the segment of \( G \) that contains \( x \). Clearly, \( \overline{C}^{[w_1,v]} \) contains at least one vertex, because \( \overline{C}^{[v,w_1]} \) contains at least three vertices.

Let \( R_1 := P^{[v_1,x]} \cup Q^{[x,w_1]} \cup \overline{C}^{[w_1,v]} \cup P^{[v_2,v]} \). Since \( (P^{[v_1,x]} \cup P^{[v_2,v]} ) \cap Q^{[x,w_1]} = \emptyset \) and all these three paths are internally vertex-disjoint from \( \overline{C}^{[w_1,v]} \), \( R_1 \) is indeed a path.

We now define a walk \( R_2 \) as follows:

- If \( R^{[v,w_1]} \cap Q'' = \emptyset \), let \( R_2 := Q^{[u_1,x]} \cup P^{[x,v]} \cup R^{[v,w_1]} \cup Q^{[w_1,w_2]} \cup Q''^{[w_2,u_2]} \).
- If \( R^{[v,w]} \cap Q'' \neq \emptyset \) (in this case, \( w_2 = w_1 \) is adjacent to \( v \) in \( G \)), let \( R_2 := Q^{[u_1,x]} \cup P^{[x,v]} \cup \overline{C}^{[w_1,v]} \cup Q''^{[w_2,u_2]} \).

Note that \( Q^{[w_1,w_2]} \cup Q''^{[w_2,u_2]} = Q^{[w_1,u_2]} \), which are written separately in the definition of \( R_2 \) for the purpose of emphasizing their locations. Since \( (Q^{[u_1,x]} \cup Q^{[w_1,u_2]} ) \cap R^{[v,w_1]} = \emptyset \) in the first case and \( (Q^{[u_1,x]} \cup Q^{[w_1,u_2]} ) \cap \overline{C}^{[w_1,v]} = \emptyset \) in the second case, these corresponding paths are internally vertex-disjoint from \( P^{[w_1,x]} \). \( R_2 \) is also a path.

By summing the lengths of \( R_1 \) and \( R_2 \), we get

\[ |R_1| + |R_2| \geq |P| + |Q| + |C| > |P| + |Q|, \]

since \( |C| \geq 3 \), which gives a contradiction to the assumption that both \( P \) and \( Q \) are longest paths. \( \square \)

This completes our second proof of Theorem 13.

\[ \square \]

\section*{6 Algorithmic remarks}

For any hereditary class of graphs for which there is a polynomial-time algorithm that computes (the length of) a longest path, it is easy to derive a polynomial-time algorithm that finds all Gallai vertices. Indeed, one just has to compute the length \( L \) of a longest path in the given (connected) graph \( G \), and then to check, for each vertex \( v \), whether the length of a longest path in \( G - v \) remains the same. If not, \( v \) is a Gallai vertex.

It is a well-known result that one can use dynamic programming to solve many combinatorial problems on graphs of bounded treewidth in polynomial or even linear time \([1, 4]\). In particular, Bodlaender \([5, \text{Thm. } 2.2]\) claims a linear-time algorithm following these lines to find a longest path in a graph with bounded treewidth. (See also \([6]\) on how to obtain in linear time a tree decomposition for graphs with bounded treewidth.) Therefore, using the idea described in the previous paragraph, one can find all Gallai vertices in time quadratic in the number of vertices of the given connected series-parallel graph.
In fact, one can do better by applying the same strategy used to compute the length of a longest path in a partial $k$-tree, but carrying more information during the process. Given a connected graph $G$ of treewidth $k$, compute in linear time a “nice” tree decomposition for $G$ (as done in [9] for instance). Then run a dynamic programming algorithm on top of this tree decomposition, to compute the length $L$ of a longest path in $G$. Roughly speaking, this algorithm computes the length of longest parts of paths within the subgraph induced by the vertices in clusters already traversed of the tree decomposition, and puts together this information while going through the tree decomposition. Specifically, when visiting a node $u$ of the tree decomposition, if $H_u$ is the subgraph of $G$ induced by the vertices in the cluster $X_u$ or clusters of nodes below $u$, for each different way that a path can behave in the cluster $X_u$ and in $H_u$, we have a configuration as the ones described in Figure 12 for the case in which $X_u$ has three vertices.

![Figure 12: A sample of the configurations for a cluster with three vertices. The whole set of configurations has to consider the labels of the vertices in the cluster.](image)

The number of such configurations depends only on the treewidth. For each such configuration, the algorithm computes the length of a longest part of a path in $H_u$ that “agrees” with that configuration. It does this using dynamic programming, that is, it computes such length for a node $u$ and one of the configurations using the information that it already computed for the children of $u$ in the tree. Some of the configurations of the children, together with new edges within $X_u$, combine into each configuration for $u$. The combinations that give raise to the longest parts are the ones of interest, and give the length of a longest part for that configuration for $u$.

In a first traversal of the tree decomposition, the value of $L$ is computed. Now, as it is usual in dynamic programming, in a reverse traversal of the tree, retracing backwards what was done to find out $L$, one can mark, for each node and each configuration, if that configuration at that node gives raise to a path of length $L$ in $G$. Once this is done for a node, the algorithm checks whether the configurations for that node that give raise to a longest path all contain one of the vertices in the cluster of that node. If so, this is a Gallai vertex. Otherwise the algorithm proceeds to the next node in the reverse traversal.

This process finishes with a Gallai vertex as long as the graph has one such vertex. In particular, for partial 2-trees, this process will find a Gallai vertex in the reverse traversal as soon as it reaches the first cluster of the tree that contains a Gallai vertex. By proceeding with the reverse traversal in this way, one can find all Gallai vertices. For bounded $k$, the running time of this algorithm is linear in the number of vertices of the graph. (Note that the number of edges in a partial $k$-tree is at most $kn$, where $n$ is the number of vertices in the partial $k$-tree.) Indeed, first computing a nice tree decomposition can be done in linear time. Second, the number of configurations depends only on $k$, and the processing of each node of the tree decomposition depends only on the number of configurations (and on the size of the cluster, which is bounded by $k + 1$ and thus also by the number of configurations). Therefore, for series-parallel graphs,
this algorithm finds a Gallai vertex (or even all Gallai vertices) in time that is linear in the number of vertices of the graph.

7 Related results and open questions

There are several questions related to Gallai’s original question that remain open. For instance, it was asked [16, 31] whether there is a vertex common to all longest paths in all 4-connected graphs. This problem is open so far, and even the more general question for \(k\)-connected graphs with larger \(k\) has not been answered. There are 3-connected examples known for which Gallai’s question has a negative answer [12].

In [10], where a proof that all 2-trees have nonempty intersection of all longest paths was presented, it was asked whether the same holds for \(k\)-trees with larger values of \(k\). As far as we know, this also has not yet been answered. In the present paper, we have proven that all connected subgraphs of 2-trees have nonempty intersection of all longest paths. We observe that the same does not hold for all subgraphs of 3-trees. Indeed, the counterexample by Walther, Voss, and Zamfirescu in [28, 30] is a connected spanning subgraph of a 3-tree (see Figure 13).

![Figure 13: The counterexample of Walther, Voss, and Zamfirescu as a subgraph of a 3-tree. Underlying edges are dotted. The number next to each vertex indicates the sequence in which they are added to the 3-tree.](image)

In other words, Gallai’s question has a positive answer for connected graphs with treewidth at most 2 (series-parallel graphs), but a negative answer for connected graphs with treewidth at most 3. As series-parallel graphs are the class of \(K_4\)-minor-free graphs, one might also ask whether the answer is positive for all (connected) \(K_5\)-minor-free graphs, but there are planar counterexamples known [24].

As split graphs and 2-trees are chordal, a natural question raised by Balister et al. [3] is whether all longest paths share a vertex in all chordal graphs. Recently, Michel Habib (personal communication) suggested that the answer to Gallai’s question might be positive in co-comparability graphs. For this class of graphs, as well as for series-parallel graphs, there is a polynomial-time algorithm to compute a longest path [14]. (For chordal graphs, computing a longest path is NP-hard [19].)

As already stated in Section 1, instead of looking at the intersection of all longest paths, Zamfirescu asked whether any \(p\) longest paths in an arbitrary connected graph contain a common
This is certainly true for \( p = 2 \), proven to be false \([21, 23]\) for \( p \geq 7 \), but still open for \( p \) in \( \{3, 4, 5, 6\} \).

References


