(1) Let $R$ be a ring with identity, $S$ the ring of all $n \times n$ matrices with entries in $R$, and $J$ a subset of $S$. Prove that $J$ is a two-sided ideal of $S$ if and only if $J$ equals the set of all $n \times n$ matrices with entries in a two-sided ideal $I$ of $R$.

(2) Let $E$ be the splitting field of $X^4 + 1$ over $\mathbb{Q}$. Compute the Galois group of $E$ over $\mathbb{Q}$. How many intermediate fields $K$ (with $\mathbb{Q} \subseteq K \subseteq E$) are normal extensions of $\mathbb{Q}$? Please explain your answer fully.

(3) Show there is no simple group of order 48.

(4) Let $M$ be the quotient of $\mathbb{Z}^4$ modulo the $\mathbb{Z}$-submodule generated by the column vectors of the following $4 \times 5$ matrix
\[
A = \begin{pmatrix}
0 & 12 & 12 & 0 & 48 \\
-12 & 12 & 0 & -24 & 12 \\
24 & -12 & 48 & 48 & 60 \\
12 & -12 & 36 & 24 & 24
\end{pmatrix}.
\]

Find the Smith normal form of $A$. Then express $M$ as a direct sum of cyclic $\mathbb{Z}$-modules. What are the free rank, the invariant factors, and the elementary divisors of $M$?

(5) (i) Let $k \subseteq K$ be a field extension. Assume that every polynomial irreducible in $k[X]$ is irreducible in $K[X]$. Show that $k$ is algebraically closed in $K$.
(ii) Show that any algebraic field extension of a perfect field is perfect.

(6) Let $R$ be a (commutative) domain. Show that the intersection of all maximal ideals of $R[X]$ is the zero ideal.