Problems for the Qualifying Examination in Algebra at GSU
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(1) Let $G$ be a group, $H \leq G$ such that $|H| < \infty$, and $P$ is a Sylow $p$-subgroup of $H$ (with $p$ a prime divisor of $|H|$). Denote by $N(H)$ and $N(P)$ the normalizers of $H$ and $P$ in $G$ respectively.

(a) Prove that $N(H) \leq N(P)$ if $P$ is normal in $H$.
(b) Prove that $HN(P) = G$ if $H$ is normal in $G$.

(2) Let $R$ be a commutative non-zero ring with unity. Assume that every ideal $I$ of $R$, with $I \neq R$, is prime. Show that $R$ is a field.

(3) Consider the polynomial $f(x) = x^4 - 4x^2 + 1 \in \mathbb{Q}[x]$. You may use the fact that $f(x)$ is irreducible over $\mathbb{Q}$ without a proof.

(a) Prove that $\mathbb{Q}(\sqrt{2} + \sqrt{3})$ is a splitting field of $f(x)$ over $\mathbb{Q}$.
(b) Determine the structure of the Galois group of $\mathbb{Q}(\sqrt{2} + \sqrt{3})$ over $\mathbb{Q}$, with explanations.

(4) Consider the integer $57575 = 5^2 \cdot 7^2 \cdot 47$.

(a) Prove or disprove: There exists a group of order 57575 that is not abelian.
(b) Classify all groups of order 57575.

(5) Let $F \subseteq K$ be an algebraic extension of fields. Prove that the following statements are equivalent:

(a) Every separable polynomial in $F[x]$ that is irreducible over $F$ remains irreducible over $K$.
(b) $F$ is separably closed in $K$ (i.e., no element $a \in K \setminus F$ is separable over $F$).

Hints: A separable polynomial is a polynomial with no repeated roots. For one direction, you might want to work in a (properly chosen) extension field $L$ of $K$. You may quote the fact that the separable closure of $F$ in $L$, defined as $\{ a \in L | a$ separable over $F \}$, is a field. Viete’s formula (describing relation between roots and coefficients) could help too.

(6) Let $R = \mathbb{Q}[x]$ and $M$ be the quotient of $R^3$ modulo the $R$-submodule generated by the columns of the $3 \times 3$ matrix $xI - A$ where $I$ is the $3 \times 3$ identity matrix and

$$A = \begin{pmatrix} 2 & 2 & 3 \\ -1 & -1 & -3 \\ 1 & 2 & 4 \end{pmatrix}.$$ 

(a) Find the Smith normal form of $xI - A$ over $R$.
(b) Up to isomorphism, express $M$ as a direct sum of cyclic $R$-modules.
(c) What are the free rank, the invariant factors, and the elementary divisors of $M$ over $R$?
(d) Determine the rational canonical form and the Jordan canonical form of $A$ over $\mathbb{Q}$. 