Please write your name on every page. Show your work.

(1) Let $k$ be a field and $k(X) = \{ f(X)/g(X) : f(X) \in k[X], 0 \neq g(X) \in k[X] \}$. Show that $k(X)$ is not an algebraically closed field. Also, find a strict subfield of $k(X)$ isomorphic to $k(X)$.

(2) Let $E \subseteq \mathbb{C}$ such that $E$ is the splitting field of $(x^2 - 3)(x^2 - 5)$ over $\mathbb{Q}$.
   (a) Find the Galois group of $E$ over $\mathbb{Q}$.
   (b) Prove $E = \mathbb{Q}[2\sqrt{3} + \sqrt{5}]$.

(3) Consider the integer $5929 = 7^2 \cdot 11^2$.
   (a) Prove or disprove: Every group of order 5929 is abelian.
   (b) Classify all groups of order 5929.

(4) Let $R = \mathbb{Q}[x]$ and $M$ be the quotient of $R^3$ modulo the $R$-submodule generated by the columns of the $3 \times 3$ matrix $xI - A$ where $I$ is the $3 \times 3$ identity matrix and

$$A = \begin{pmatrix} -9 & -10 & -1 \\ 7 & 8 & 1 \\ 3 & 2 & -1 \end{pmatrix}.$$ 

   (a) Find the Smith normal form of $xI - A$ over $R$.
   (b) Up to isomorphism, express $M$ as a direct sum of cyclic $R$-modules.
   (c) What are the free rank, the invariant factors, and the elementary divisors of $M$ over $R$?
   (d) Determine the rational canonical form and the Jordan canonical form of $A$ over $\mathbb{Q}$.

(5) Let $G$ be a finite group and $H$ a subgroup of $G$ with $H \neq G$. Prove $G \neq \bigcup_{g \in G} (gHg^{-1})$.

(6) Let $A$ be a unique factorization domain. Show that any minimal nonzero prime ideal of $A$ is principal. (An ideal $P$ is called a minimal nonzero prime ideal if $P$ is nonzero prime ideal such that there is no nonzero prime ideal strictly contained in $P$.)