Minimum codegree threshold for Hamilton $\ell$-cycles in $k$-uniform hypergraphs

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1. Introduction

A well-known result of Dirac [4] states that every graph $G$ on $n \geq 3$ vertices with minimum degree $\delta(G) \geq n/2$ contains a Hamilton cycle. In recent years, researchers have worked on extending this result to hypergraphs — see recent surveys [15,18]. Given $k \geq 2$, a $k$-uniform hypergraph (in short, $k$-graph) consists of a vertex set $V$ and an edge set $E \subseteq \binom{V}{k}$, where every edge is a $k$-element subset of $V$. Given a $k$-graph $\mathcal{H}$ with a set $S$ of $d$ vertices (where $1 \leq d \leq k - 1$) we define $\deg_\mathcal{H}(S)$ to be the number of edges containing $S$ (the subscript $\mathcal{H}$ is omitted if it is clear from the context). The minimum
Theorem 1.1. (See [7].) Fix integers \( k \geq 3 \) and \( 1 \leq \ell < k/2 \). Assume that \( \gamma > 0 \) and \( n \in (k-\ell)\mathbb{N} \) is sufficiently large. If \( \mathcal{H} = (V, E) \) is a \( k \)-graph on \( n \) vertices such that \( \delta_{k-1}(\mathcal{H}) \geq \frac{1}{2(k-\ell)} + \gamma \), then \( \mathcal{H} \) contains a Hamilton \( \ell \)-cycle.

Later Kühn, Mycroft, and Osthus [13] proved that whenever \( k - \ell \) does not divide \( k \), every \( k \)-graph on \( n \) vertices with \( \delta_{k-1}(\mathcal{H}) \geq \frac{n}{2(k-\ell)} + o(n) \) contains a Hamilton \( \ell \)-cycle. This generalizes Theorem 1.1 because \( \left\lfloor \frac{k}{k-\ell} \right\rfloor = 2 \) when \( k < 2/2 \). Rödl and Ruciński [18, Problem 2.9] asked for the exact minimum codegree threshold for Hamilton \( \ell \)-cycles in \( k \)-graphs. The \( k = 3 \) and \( \ell = 1 \) case was answered by Czygrinow and Molla [3] recently. In this paper we determine this threshold for all \( k \geq 3 \) and \( \ell < k/2 \).

Theorem 1.2 (Main result). Fix integers \( k \geq 3 \) and \( 1 \leq \ell < k/2 \). Assume that \( n \in (k-\ell)\mathbb{N} \) is sufficiently large. If \( \mathcal{H} = (V, E) \) is a \( k \)-graph on \( n \) vertices such that

\[
\delta_{k-1}(\mathcal{H}) \geq \frac{n}{2(k-\ell)},
\]

then \( \mathcal{H} \) contains a Hamilton \( \ell \)-cycle.
The following simple construction [13, Proposition 2.2] shows that Theorem 1.2 is best possible, and the aforementioned results in [7,12–14] are asymptotically best possible. Let $\mathcal{H}_0 = (V,E)$ be an $n$-vertex $k$-graph in which $V$ is partitioned into sets $A$ and $B$ such that $|A| = \lceil \frac{n}{k-\ell} \rceil - 1$. The edge set $E$ consists of all $k$-sets that intersect $A$. It is easy to see that $\delta_{k-1}(\mathcal{H}_0) = |A|$. However, an $\ell$-cycle on $n$ vertices has $n/(k-\ell)$ edges and every vertex on such a cycle lies in at most $\lceil \frac{k}{k-\ell} \rceil$ edges. Since $\lceil \frac{k}{k-\ell} \rceil |A| < n/(k-\ell)$, $\mathcal{H}_0$ contains no Hamilton $\ell$-cycle.

A related problem was studied by Buß, Hán, and Schacht [1], who proved that every 3-graph $\mathcal{H}$ on $n$ vertices with minimum vertex degree $\delta_1(\mathcal{H}) \geq (\frac{7}{16} + o(1))(\frac{n}{2})$ contains a loose Hamilton cycle. Recently we [10] improved this to an exact result.

Using the typical approach of obtaining exact results, our proof of Theorem 1.2 consists of an extremal case and a nonextremal case.

**Definition 1.3.** Let $\Delta > 0$, a $k$-graph $\mathcal{H}$ on $n$ vertices is called $\Delta$-extremal if there is a set $B \subset V(\mathcal{H})$, such that $|B| = \lceil \frac{2(k-\ell)-1}{2(k-\ell)}n \rceil$ and $e(B) \leq \Delta n^k$.

**Theorem 1.4 (Nonextremal case).** For any integer $k \geq 3$, $1 \leq \ell < k/2$ and $0 < \Delta < 1$ there exists $\gamma > 0$ such that the following holds. Suppose that $\mathcal{H}$ is a $k$-graph on $n$ vertices such that $n \in (k-\ell)\mathbb{N}$ is sufficiently large. If $\mathcal{H}$ is not $\Delta$-extremal and satisfies $\delta_{k-1}(\mathcal{H}) \geq (\frac{1}{2(k-\ell)} - \gamma)n$, then $\mathcal{H}$ contains a Hamilton $\ell$-cycle.

**Theorem 1.5 (Extremal case).** For any integer $k \geq 3$, $1 \leq \ell < k/2$ there exists $\Delta > 0$ such that the following holds. Suppose $\mathcal{H}$ is a $k$-graph on $n$ vertices such that $n \in (k-\ell)\mathbb{N}$ is sufficiently large. If $\mathcal{H}$ is $\Delta$-extremal and satisfies (1.1), then $\mathcal{H}$ contains a Hamilton $\ell$-cycle.

Theorem 1.2 follows from Theorem 1.4 and 1.5 immediately by choosing $\Delta$ from Theorem 1.5.

Let us compare our proof with those in the aforementioned papers. There is no extremal case in [7,12–14] because only asymptotic results were proved. Our Theorem 1.5 is new and more general than [3, Theorem 3.1]. Following previous work [7,13,19–21], we prove Theorem 1.4 by using the absorbing method initiated by Rödl, Ruciński and Szemerédi. More precisely, we find the desired Hamilton $\ell$-cycle by applying the Absorbing Lemma (Lemma 2.1), the Reservoir Lemma (Lemma 2.2), and the Path-cover Lemma (Lemma 2.3). In fact, when $\ell < k/2$, the Absorbing Lemma and the Reservoir Lemma are not very difficult and already proven in [7] (in contrast, when $\ell > k/2$, the Absorbing Lemma in [13] is more difficult to prove). Thus the main step is to prove the Path-cover Lemma. As shown in [7,13], after the Regularity Lemma is applied, it suffices to prove that the cluster $k$-graph $\mathcal{K}$ can be tiled almost perfectly by the $k$-graph $\mathcal{F}_{k,\ell}$, whose vertex set consists of disjoint sets $A_1, \ldots, A_{a-1}, B$ of size $k-1$, and whose edges are all the $k$-sets of the form $A_i \cup \{b\}$ for $i = 1, \ldots, a-1$ and all $b \in B$, where $a = \lceil \frac{k}{k-\ell} \rceil (k-\ell)$. In this paper we reduce the problem to tile $\mathcal{K}$ with a much simpler $k$-graph $\mathcal{Y}_{k,2\ell}$, which
consists of two edges sharing $2\ell$ vertices. Because of the simple structure of $\mathcal{Y}_{k,2\ell}$, we can easily find an almost perfect $\mathcal{Y}_{k,2\ell}$-tiling unless $K$ is in the extremal case (thus the original $k$-graph $\mathcal{H}$ is in the extremal case). Interestingly $\mathcal{Y}_{3,2}$-tiling was studied in the very first paper [14] on loose Hamilton cycles but as a separate problem. Our recent paper [10] indeed used $\mathcal{Y}_{3,2}$-tiling as a tool to prove the corresponding path-cover lemma. On the other hand, the authors of [3] used a different approach (without the Regularity Lemma) to prove the Path-tiling Lemma (though they did not state such lemma explicitly).

The rest of the paper is organized as follows. We prove Theorem 1.4 in Section 2 and Theorem 1.5 in Section 3, and give concluding remarks in Section 4.

**Notation.** Given an integer $k \geq 0$, a $k$-set is a set with $k$ elements. For a set $X$, we denote by $\binom{X}{k}$ the family of all $k$-subsets of $X$. Given a $k$-graph $\mathcal{H}$ and a set $A \subseteq V(\mathcal{H})$, we denote by $e_\mathcal{H}(A)$ the number of the edges of $\mathcal{H}$ in $A$. We often omit the subscript that represents the underlying hypergraph if it is clear from the context. Given a $k$-graph $\mathcal{H}$ with two vertex sets $S, R$ such that $|S| < k$, we denote by $\deg_\mathcal{H}(S, R)$ the number of $(k - |S|)$-sets $T \subseteq R$ such that $S \cup T$ is an edge of $\mathcal{H}$ (in this case, $T$ is called a neighbor of $S$). We define $\overline{\deg}_\mathcal{H}(S, R) = \binom{|R|-|S|}{k-|S|} - \deg(S, R)$ as the number of non-edges on $S \cup R$ that contain $S$. When $R = V(\mathcal{H})$ (and $\mathcal{H}$ is obvious), we simply write $\deg(S)$ and $\overline{\deg}(S)$. When $S = \{v\}$, we use $\deg(v, R)$ instead of $\deg(\{v\}, R)$.

A $k$-graph $\mathcal{P}$ is an $\ell$-path if there is an ordering $(v_1, \ldots, v_\ell)$ of its vertices such that every edge consists of $k$ consecutive vertices and two consecutive edges intersect in exactly $\ell$ vertices. Note that this implies that $k - \ell$ divides $t - \ell$. In this case, we write $\mathcal{P} = v_1 \cdot \cdot \cdot v_\ell$ and call two $\ell$-sets $\{v_1, \ldots, v_\ell\}$ and $\{v_{\ell-\ell+1}, \ldots, v_\ell\}$ ends of $\mathcal{P}$.

### 2. Proof of Theorem 1.4

In this section we prove Theorem 1.4 by following the approach in [7].

2.1. **Auxiliary lemmas and proof of Theorem 1.4**

We need [7, Lemma 5] and [7, Lemma 6] of Hán and Schacht, in which only a linear codegree condition is needed. Given a $k$-graph $\mathcal{H}$ with an $\ell$-path $\mathcal{P}$ and a vertex set $U \subseteq V(\mathcal{H}) \setminus V(\mathcal{P})$ with $|U| \in (k - \ell)\mathbb{N}$, we say that $\mathcal{P}$ absorbs $U$ if there exists an $\ell$-path $Q$ of $\mathcal{H}$ with $V(Q) = V(\mathcal{P}) \cup U$ such that $\mathcal{P}$ and $Q$ have exactly the same ends.

**Lemma 2.1** (Absorbing Lemma). (See [7].) For all integers $k \geq 3$ and $1 \leq \ell < k/2$ and every $\gamma_1 > 0$ there exist $\eta > 0$ and an integer $n_0$ such that the following holds. Let $\mathcal{H}$ be a $k$-graph on $n \geq n_0$ vertices with $\delta_{k-1}(\mathcal{H}) \geq \gamma_1 n$. Then $\mathcal{H}$ contains an absorbing $\ell$-path $\mathcal{P}$ with $|V(\mathcal{P})| \leq \gamma_1^2 n$ that can absorb any subset $U \subset V(\mathcal{H}) \setminus V(\mathcal{P})$ of size $|U| \leq \eta n$ and $|U| \in (k - \ell)\mathbb{N}$. 
Lemma 2.2 (Reservoir Lemma). (See [7].) For all integers \( k \geq 3 \) and \( 1 \leq \ell < k/2 \) and every \( 0 < d, \gamma_2 < 1 \) there exists an \( n_0 \) such that the following holds. Let \( \mathcal{H} \) be a \( k \)-graph on \( n > n_0 \) vertices with \( \delta_{k-1}(\mathcal{H}) \geq dn \), then there is a set \( R \) of size at most \( \gamma_2 n \) such that for all \((k-1)\)-sets \( S \in \binom{V}{k-1} \) we have \( \deg(S, R) \geq d \gamma_2 n/2 \).

The main step in our proof of Theorem 1.4 is the following lemma, which is stronger than [7, Lemma 7]. We defer its proof to the next subsection.

Lemma 2.3 (Path-cover Lemma). For all integers \( k \geq 3, 1 \leq \ell < k/2 \), and every \( \gamma_3, \alpha > 0 \) there exist integers \( p \) and \( n_0 \) such that the following holds. Let \( \mathcal{H} \) be a \( k \)-graph on \( n > n_0 \) vertices with \( \delta_{k-1}(\mathcal{H}) \geq \left( \frac{1}{2(k-\ell)} - \gamma_3 \right) n \), then there is a family of at most \( p \) vertex disjoint \( \ell \)-paths that together cover all but at most \( \alpha n \) vertices of \( \mathcal{H} \), or \( \mathcal{H} \) is \( 14 \gamma_3 \)-extremal.

We can now prove Theorem 1.4 in a similar way as in [7].

Proof of Theorem 1.4. Given \( k \geq 3, 1 \leq \ell < k/2 \) and \( 0 < \Delta < 1 \), let \( \gamma = \min\{ \frac{\Delta}{43}, \frac{1}{4k^2} \} \) and \( n \in (k-\ell)\mathbb{N} \) be sufficiently large. Suppose that \( \mathcal{H} = (V, E) \) is a \( k \)-graph on \( n \) vertices with \( \delta_{k-1}(\mathcal{H}) \geq \left( \frac{1}{2(k-\ell)} - \gamma \right) n \). Since \( \frac{1}{2(k-\ell)} - \gamma > \gamma \), we can apply Lemma 2.1 with \( \gamma_1 = \gamma \) and obtain \( \eta > 0 \) and an absorbing path \( P_0 \) with ends \( S_0, T_0 \) such that \( |V(P_0)| \leq \gamma^5 n \) and \( P_0 \) can absorb any \( u \) vertices outside \( P_0 \) if \( u \leq \eta n \) and \( u \in (k-\ell)\mathbb{N} \).

Let \( V_1 = (V \setminus V(P_0)) \cup S_0 \cup T_0 \) and \( \mathcal{H}_1 = \mathcal{H}[V_1] \). Note that \( |V(P_0)| \leq \gamma^5 n \) implies that \( \delta_{k-1}(\mathcal{H}_1) \geq \left( \frac{1}{2(k-\ell)} - \gamma \right)n - \gamma^5 n \geq \frac{1}{2k} n \) as \( \gamma < \frac{1}{4k^2} \) and \( \ell \geq 1 \). We next apply Lemma 2.2 with \( d = \frac{1}{2k} \) and \( \gamma_2 = \min\{ \eta/2, \gamma \} \) to \( \mathcal{H}_1 \) and get a reservoir \( R \subset V_1 \) with \( |R| \leq \gamma_2 |V(\mathcal{H}_1)| \leq \gamma_2 n \) such that for any \((k-1)\)-set \( S \subset V_1 \), we have

\[
\deg(S, R) \geq d \gamma_2 |V_1|/2 \geq d \gamma_2 n/4. \tag{2.1}
\]

Let \( V_2 = V \setminus (V(P_0) \cup R) \), \( n_2 = |V_2| \), and \( \mathcal{H}_2 = \mathcal{H}[V_2] \). Note that \( |V(P_0) \cup R| \leq \gamma_1^5 n + \gamma_2 n \leq 2 \gamma n \), so

\[
\delta_{k-1}(\mathcal{H}_2) \geq \left( \frac{1}{2(k-\ell)} - \gamma \right)n - 2 \gamma n \geq \left( \frac{1}{2(k-\ell)} - 3 \gamma \right)n_2.
\]

Applying Lemma 2.3 to \( \mathcal{H}_2 \) with \( \gamma_3 = 3 \gamma \) and \( \alpha = \eta/2 \), we obtain at most \( p \) vertex disjoint \( \ell \)-paths that cover all but at most \( \alpha n_2 \) vertices of \( \mathcal{H}_2 \), unless \( \mathcal{H}_2 \) is \( 14 \gamma_3 \)-extremal. In the latter case, there exists \( B' \subset V_2 \) such that \( |B'| = \left[ \frac{2k-2\ell-1}{2k} n_2 \right] \) and \( e(B') \leq 42 \gamma n_2^k \). Then we add at most \( n - n_2 \leq 2 \gamma n \) vertices from \( V \setminus B' \) to \( B' \) and obtain a vertex set \( B \subset V(\mathcal{H}) \) such that \( |B| = \left[ \frac{2k-2\ell-1}{2k} n \right] \) and

\[
e(B) \leq 42 \gamma n_2^k + 2 \gamma n \cdot \left( \frac{n - 1}{k - 1} \right) \leq 42 \gamma n^k + \gamma n^k \leq \Delta n^k,
\]

which means that \( \mathcal{H} \) is \( \Delta \)-extremal, a contradiction. In the former case, denote these \( \ell \)-paths by \( \{P_i\}_{i \in [p']} \) for some \( p' \leq p \), and their ends by \( \{S_i, T_i\}_{i \in [p']} \). Note that
both $S_i$ and $T_i$ are $\ell$-sets for $\ell < k/2$. We arbitrarily pick disjoint $(k - 2\ell - 1)$-sets $X_0, X_1, \ldots, X_{p'} \subset R \setminus (S_0 \cup T_0)$ (note that $k - 2\ell - 1 \geq 0$). Let $T_{p'+1} = T_0$. By (2.1), as $d\gamma_2n/4 \geq k(p' + 1)$, we may find $p' + 1$ vertices $v_0, v_1, \ldots, v_{p'} \in R$ such that $S_i \cup T_{i+1} \cup X_i \cup \{v_i\} \in E(H)$ for $0 \leq i \leq p'$. We thus connect $P_0, P_1, \ldots, P_{p'}$ together and obtain an $\ell$-cycle $C$. Note that

$$|V(H) \setminus V(C)| \leq |R| + \alpha n_2 \leq \gamma_2n + \alpha n \leq \eta n$$

and $k - \ell$ divides $|V \setminus V(C)|$ because $k - \ell$ divides both $n$ and $|V(C)|$. So we can use $P_0$ to absorb all unused vertices in $R$ and uncovered vertices in $V_2$ thus obtaining a Hamilton $\ell$-cycle in $H$. $\square$

The rest of this section is devoted to the proof of Lemma 2.3.

2.2. Proof of Lemma 2.3

Following the approach in [7], we use the Weak Regularity Lemma, which is a straightforward extension of Szemerédi’s regularity lemma for graphs [22].

Let $H = (V, E)$ be a $k$-graph and let $A_1, \ldots, A_k$ be mutually disjoint non-empty subsets of $V$. We define $e(A_1, \ldots, A_k)$ to be the number of crossing edges, namely, those with one vertex in each $A_i$, $i \in [k]$, and the density of $H$ with respect to $(A_1, \ldots, A_k)$ as

$$d(A_1, \ldots, A_k) = \frac{e(A_1, \ldots, A_k)}{|A_1| \cdots |A_k|}.$$

We say a $k$-tuple $(V_1, \ldots, V_k)$ of mutually disjoint subsets $V_1, \ldots, V_k \subseteq V$ is $(\epsilon, d)$-regular, for $\epsilon > 0$ and $d \geq 0$, if

$$|d(A_1, \ldots, A_k) - d| \leq \epsilon$$

for all $k$-tuples of subsets $A_i \subseteq V_i$, $i \in [k]$, satisfying $|A_i| \geq \epsilon |V_i|$. We say $(V_1, \ldots, V_k)$ is $\epsilon$-regular if it is $(\epsilon, d)$-regular for some $d \geq 0$. It is immediate from the definition that in an $(\epsilon, d)$-regular $k$-tuple $(V_1, \ldots, V_k)$, if $V'_i \subset V_i$ has size $|V'_i| \geq c|V_i|$ for some $c \geq \epsilon$, then $(V'_1, \ldots, V'_k)$ is $(\epsilon/c, d)$-regular.

**Theorem 2.4** (Weak Regularity Lemma). Given $t_0 \geq 0$ and $\epsilon > 0$, there exist $T_0 = T_0(t_0, \epsilon)$ and $n_0 = n_0(t_0, \epsilon)$ so that for every $k$-graph $H = (V, E)$ on $n > n_0$ vertices, there exists a partition $V = V_0 \cup V_1 \cup \cdots \cup V_t$ such that

1. $t_0 \leq t \leq T_0$,
2. $|V_1| = |V_2| = \cdots = |V_t|$ and $|V_0| \leq \epsilon n$,
3. for all but at most $\epsilon(t \choose k)$ $k$-subsets $\{i_1, \ldots, i_k\} \subset [t]$, the $k$-tuple $(V_{i_1}, \ldots, V_{i_k})$ is $\epsilon$-regular.
The partition given in Theorem 2.4 is called an \( \epsilon \)-regular partition of \( \mathcal{H} \). Given an \( \epsilon \)-regular partition of \( \mathcal{H} \) and \( d \geq 0 \), we refer to \( V_i \), \( i \in [t] \), as clusters and define the cluster hypergraph \( \mathcal{K} = \mathcal{K}(\epsilon, d) \) with vertex set \([t]\) and \( \{i_1, \ldots, i_k\} \subseteq [t] \) is an edge if and only if \((V_i, \ldots, V_{i_k}) \) is \( \epsilon \)-regular and \( d(V_i, \ldots, V_{i_k}) \geq d \).

We combine Theorem 2.4 and [7, Proposition 16] into the following corollary, which shows that the cluster hypergraph almost inherits the minimum degree of the original hypergraph. Its proof is standard and similar as the one of [7, Proposition 16] so we omit it.\(^1\)

**Corollary 2.5.** (See [7].) Given \( c, \epsilon, d > 0 \), integers \( k \geq 3 \) and \( t_0 \), there exist \( T_0 \) and \( n_0 \) such that the following holds. Let \( \mathcal{H} \) be a \( k \)-graph on \( n > n_0 \) vertices with \( \delta_{k-1}(\mathcal{H}) \geq cn \). Then \( \mathcal{H} \) has an \( \epsilon \)-regular partition \( V_0 \cup V_1 \cup \cdots \cup V_t \) with \( t_0 \leq t \leq T_0 \), and in the cluster hypergraph \( \mathcal{K} = \mathcal{K}(\epsilon, d) \), all but at most \( \sqrt{\epsilon}t^{k-1} \) \((k-1)\)-subsets \( S \) of \([t]\) satisfy \( \deg_{\mathcal{K}}(S) \geq (c - d - \sqrt{\epsilon})t - (k - 1) \).

Let \( \mathcal{H} \) be a \( k \)-partite \( k \)-graph with partition classes \( V_1, \ldots, V_k \). Given \( 1 \leq \ell < k/2 \), we call an \( \ell \)-path \( \mathcal{P} \) with edges \( \{e_1, \ldots, e_q\} \) canonical with respect to \((V_1, \ldots, V_k)\) if

\[
e_i \cap e_{i+1} = \bigcup_{j \in [\ell]} V_j \quad \text{or} \quad e_i \cap e_{i+1} \subseteq \bigcup_{j \in [2\ell]\setminus[t]} V_j
\]

for \( i \in [q-1] \). When \( j > 2\ell \), all \( e_1 \cap V_j, \ldots, e_q \cap V_j \) are distinct and thus \( |V(\mathcal{P}) \cap V_j| = |(e_1 \cup \cdots \cup e_q) \cap V_j| = q \). When \( j \leq 2\ell \), exactly one of \( e_{i-1} \cap e_i \) and \( e_i \cap e_{i+1} \) intersects \( V_j \). Thus \( |V(\mathcal{P}) \cap V_j| = 2^{j-1}q \) if \( q \) is odd.

We need the following proposition from [7].

**Proposition 2.6.** [7, Proposition 19] Suppose that \( 1 \leq \ell < k/2 \) and \( \mathcal{H} \) is a \( k \)-partite, \( k \)-graph with partition classes \( V_1, \ldots, V_k \) such that \(|V_i| = m \) for all \( i \in [k] \), and \(|E(\mathcal{H})| \geq dn^k \). Then there exists a canonical \( \ell \)-path in \( \mathcal{H} \) with \( t > \frac{dm}{2(k-\ell)} \) edges.

In [7] the authors used Proposition 2.6 to cover an \((\epsilon, d)\)-regular tuple \((V_1, \ldots, V_k)\) of sizes \(|V_1| = \cdots = |V_{k-1}| = (2k - 2\ell - 1)m \) and \(|V_k| = (k - 1)m \) with vertex disjoint \( \ell \)-paths. Our next lemma shows that an \((\epsilon, d)\)-regular tuple \((V_1, \ldots, V_k)\) of sizes \(|V_1| = \cdots = |V_{2\ell}| = m \) and \(|V_i| = 2m \) for \( i > 2\ell \) can be covered with \( \ell \)-paths.

**Lemma 2.7.** Fix \( k \geq 3 \), \( 1 \leq \ell < k/2 \) and \( \epsilon, d > 0 \) such that \( d > 2\epsilon \). Let \( m > \frac{k\epsilon^2}{2(d-\epsilon)} \).

Suppose \( V = (V_1, V_2, \ldots, V_k) \) is an \((\epsilon, d)\)-regular \( k \)-tuple with

\[
|V_1| = \cdots = |V_{2\ell}| = m \quad \text{and} \quad |V_{2\ell+1}| = \cdots = |V_k| = 2m.
\] (2.2)

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\(^1\) Roughly speaking, the lower bound for \( \deg_{\mathcal{K}}(S) \) contains \(-d\) when forming \( \mathcal{K} \), we discard all \( k \)-tuple \((V_1, \ldots, V_k)\) of density less than \( d \) contains \(-\sqrt{\epsilon}\) because at most \( \epsilon^{(k)} \) \( k \)-tuple are not regular, and contains \(-(k-1)\) because we discard all non-crossing edges of \( \mathcal{H} \).
Then there are at most \( \frac{2k}{(d-\epsilon)e} \) vertex-disjoint \( \ell \)-paths that together cover all but at most \( 2kem \) vertices of \( V \).

**Proof.** We greedily find vertex-disjoint canonical \( \ell \)-paths of odd length by Proposition 2.6 in \( V \) until less than \( em \) vertices are uncovered in \( V_1 \) as follows. Suppose that we have obtained \( \ell \)-paths \( P_1, \ldots, P_p \) for some \( p \geq 0 \). Let \( q = \sum_{j=1}^{p} e(P_j) \). Assume that for all \( j \), \( P_j \) is canonical with respect to \( V \) and \( e(P_j) \) is odd. Then \( \bigcup_{j=1}^{p} P_i \) contains \( \frac{q+p}{2} \) vertices of \( V_i \) for \( i \in [2\ell] \) and \( q \) vertices of \( V_i \) for \( i > 2\ell \). For \( i \in [k] \), let \( U_i \) be the set of uncovered vertices of \( V_i \) and assume that \( |U_1| \geq em \). Using (2.2), we derive that 

\[
|U_{2\ell+1}| = \cdots = |U_k| = 2|U_1| + p. \tag{2.3}
\]

We now consider a \( k \)-partite subhypergraph \( V' \) with arbitrary \( |U_1| \) vertices in each \( U_i \) for \( i \in [k] \). By regularity, \( V' \) contains at least \( (d-\epsilon)|U_1|^k \) edges, so we can apply Proposition 2.6 and find an \( \ell \)-path of odd length at least \( \frac{(d-\epsilon)|U_1|^k}{2(k-\ell)} - 1 \geq \frac{(d-\epsilon)|U_1|^k}{2k} \) (dismiss one edge if needed). We continue this process until \( |U_1| < em \). Let \( P_{k-1}, \ldots, P_{k} \) be the \( \ell \)-paths obtained in \( V \) after the iteration stops. Since \( |V_1 \cap V(P_j)| \geq \frac{(d-\epsilon)|U_1|^k}{2k} \) for every \( j \), we have

\[
p \leq \frac{m}{(d-\epsilon)|U_1|^k} = \frac{2k}{(d-\epsilon)e}.
\]

Since \( m > \frac{k^2}{e(2d-\epsilon)} \), it follows that \( p(k-2\ell) < \frac{2k^2}{(d-\epsilon)e} < 2em \). By (2.3), the total number of uncovered vertices in \( V \) is

\[
\sum_{i=1}^{k} |U_i| = |U_1|2\ell + (2|U_1| + p)(k-2\ell) = 2(k-\ell)|U_1| + p(k-2\ell) < 2(k-1)em + 2em = 2kem. \quad \square
\]

Given \( k \geq 3 \) and \( 0 \leq b < k \), let \( Y_{k,b} \) be a \( k \)-graph with two edges that share exactly \( b \) vertices. In general, given two (hyper)graphs \( G \) and \( H \), a \( G \)-tiling is a sub(hyper)graph of \( H \) that consists of vertex-disjoint copies of \( G \). A \( G \)-tiling is perfect if it is a spanning sub(hyper)graph of \( H \). The following lemma is the main step in our proof of Lemma 2.3 and we prove it in the next subsection. Note that it generalizes [2, Lemma 3.1] of Czygrinow, DeBiasio, and Nagle.

**Lemma 2.8** (\( Y_{k,b} \)-tiling lemma). Given integers \( k \geq 3 \), \( 1 \leq b < k \) and constants \( \gamma, \beta > 0 \), there exist \( 0 < \epsilon' < \gamma \beta \) and an integer \( n' \) such that the following holds. Suppose \( H \) is a \( k \)-graph on \( n > n' \) vertices with \( \text{deg}(S) \geq \left( \frac{1}{2k-b} - \gamma \right)n \) for all but at most \( \epsilon'n^{k-1} \) sets \( S \in \binom{V}{k-1} \), then there is a \( Y_{k,b} \)-tiling that covers all but at most \( \beta n \) vertices of \( H \) unless \( H \) contains a vertex set \( B \) such that \( |B| = \lfloor \frac{2k-b-1}{2k-b} \rfloor n \) and \( e(B) < 6\gamma n^k \).
Now we are ready to prove Lemma 2.3.

**Proof of Lemma 2.3.** Fix integers $k, \ell, 0 < \gamma_3, \alpha < 1$. Let $\epsilon', n'$ be the constants returned from Lemma 2.8 with $b = 2\ell$, $\gamma = 2\gamma_3$, and $\beta = \alpha/2$. Thus $\epsilon' < \gamma \beta = \gamma_3 \alpha$. Let $T_0$ be the constant returned from Corollary 2.5 with $c = \frac{1}{2(k-\ell)} - \gamma_3$, $\epsilon = (\epsilon')^2/16$, $d = \gamma_3/2$ and $t_0 > \max\{n', 4k/\gamma_3\}$. Furthermore, let $p = \frac{2T_0}{(d-2\ell)\epsilon}$.

Let $n$ be sufficiently large and let $\mathcal{H}$ be a $k$-graph on $n$ vertices with $\delta_{k-1}(\mathcal{H}) \geq (\frac{1}{2(k-\ell)} - \gamma_3)n$. Applying Corollary 2.5 with the constants chosen above, we obtain an $\epsilon$-regular partition and a cluster hypergraph $\mathcal{K} = \mathcal{K}(\epsilon, d)$ on $[t]$ such that for all but at most $\sqrt{\epsilon}t^{k-1}(k-1)$-sets $S \in \binom{[t]}{k-1}$, 

$$\deg_{\mathcal{K}}(S) \geq \left(\frac{1}{2(k-\ell)} - \gamma_3 - d - \sqrt{\epsilon}\right)t - (k-1) \geq \left(\frac{1}{2(k-\ell)} - 2\gamma_3\right)t,$$

because $d = \gamma_3/2$, $\sqrt{\epsilon} = \epsilon'/4 < \gamma_3/4$ and $k-1 < \gamma_3 t_0/4 \leq \gamma_3 t/4$. Let $m$ be the size of clusters, then $(1-\epsilon)\frac{n}{t} \leq m \leq \frac{n}{t}$. Applying Lemma 2.8 with the constants chosen above, we derive that either there is a $Y_{k,2\ell}$-tiling $\mathcal{Y}$ of $\mathcal{K}$ which covers all but at most $\beta t$ vertices of $\mathcal{K}$ or there exists a set $B \subseteq V(\mathcal{K})$, such that $|B| = \lceil \frac{2k-2\ell-1}{2(k-\ell)} t \rceil$ and $e_{\mathcal{K}}(B) \leq 12\gamma_3 t^k$.

In the latter case, let $B' \subseteq V(\mathcal{H})$ be the union of the clusters in $B$. By regularity, 

$$e_{\mathcal{H}}(B') \leq e_{\mathcal{K}}(B) \cdot m^k + \left(\frac{t}{k}\right)^k \cdot d \cdot m^k + \epsilon \cdot \left(\frac{t}{k}\right)^k \cdot m^k + t \left(\frac{m}{2}\right) \left(\frac{n}{k-2}\right),$$

where the right-hand side bounds the number of edges from regular $k$-tuples with high density, edges from regular $k$-tuples with low density, edges from irregular $k$-tuples and edges that lie in at most $k-1$ clusters. Since $m \leq \frac{n}{t}$, $\epsilon < \gamma_3/16$, $d = \gamma_3/2$, and $t^{-1} < t_0^{-1} < \gamma_3/(4k)$, we obtain that 

$$e_{\mathcal{H}}(B') \leq 12\gamma_3 t^k \cdot \binom{n}{k}^{k} + \left(\frac{t}{k}\right)^k \gamma_3 \binom{n}{k} + \gamma_3 \frac{1}{16} \binom{t}{k} \binom{n}{k} + t \binom{m}{2} \binom{n}{k-2} \binom{n}{k-2} \leq 13\gamma_3 n^k.$$

Note that $|B'| = \lceil \frac{2k-2\ell-1}{2(k-\ell)} t \rceil m \leq \frac{2k-2\ell-1}{2(k-\ell)} t \cdot \frac{n}{t} = \frac{2k-2\ell-1}{2(k-\ell)} n$, and consequently $|B'| \leq \lceil \frac{2k-2\ell-1}{2(k-\ell)} n \rceil$. On the other hand,

$$|B'| = \left\lfloor \frac{2k-2\ell-1}{2(k-\ell)} t \right\rfloor m \geq \left(\frac{2k-2\ell-1}{2(k-\ell)} t - 1\right) \left(1 - \epsilon\right) \frac{n}{t}$$

$$\geq \left(\frac{2k-2\ell-1}{2(k-\ell)} t - \epsilon\right) \frac{n}{t} = \frac{2k-2\ell-1}{2(k-\ell)} n - \epsilon n.$$
By adding at most \( en \) vertices from \( V \setminus B' \) to \( B' \), we get a set \( B'' \subseteq V(\mathcal{H}) \) of size exactly 
\[
\left\lfloor \frac{2k-2\ell-1}{2(k-\ell)} \cdot n \right\rfloor,
\]
with \( e(B'') \leq e(B') + en \cdot n^{k-1} < 14\gamma_3 n^k \). Hence \( \mathcal{H} \) is \( 14\gamma_3 \)-extremal.

In the former case, let \( m' = \lceil m/2 \rceil \). If \( m \) is odd, we throw away one vertex from each cluster covered by \( \mathcal{Y} \) (we do nothing if \( m \) is even). Thus, the union of the clusters covered by \( \mathcal{Y} \) contains all but at most \( \beta tm + |V_0| + t \leq \alpha n/2 + 2\epsilon n \) vertices of \( \mathcal{H} \). We take the following procedure to each member \( \mathcal{Y}' \) with edges \( \{1, \ldots, k\} \) and \( \{k-2\ell + 1, \ldots, 2k-2\ell\} \). For \( i \in \{2k-2\ell\} \), let \( W_i \) denote the corresponding cluster in \( \mathcal{H} \). We split each \( W_i, i = k-2\ell+1, \ldots, k \), into two disjoint sets \( W_i^1 \) and \( W_i^2 \) of equal size. Each of the \( k \)-tuples \( (W_{1}^{1}, \ldots, W_{k}^{1}) \) and \( (W_{1}^{2}, \ldots, W_{k}^{2}) \) is \( (2\epsilon, d') \)-regular for some \( d' \geq d \) and of sizes \( m', \ldots, m', 2m', \ldots, 2m' \). Applying Lemma 2.7 to these two \( k \)-tuples, we find a family of at most 
\[
2 \cdot \frac{t}{2k-2\ell} \cdot \frac{\ell}{(d-2\ell)e} + \frac{\ell}{2k-2\ell} = p \text{ paths and covers all but at most}
\]
vertices of \( \mathcal{H} \), where we use \( 2k-2\ell > k \) and \( \epsilon = (\epsilon')^2/16 < (\gamma_3 \alpha)^2/16 < \alpha/12 \). This completes the proof. \( \square \)

2.3. Proof of Lemma 2.8

We first give an upper bound on the size of \( k \)-graphs containing no copy of \( \mathcal{Y}_{k,b} \). In its proof, we use the concept of link (hyper)graph: given a \( k \)-graph \( \mathcal{H} \) with a set \( S \) of at most \( k-1 \) vertices, the link graph of \( S \) is the \( (k-|S|) \)-graph with vertex set \( V(\mathcal{H}) \setminus S \) and edge set \( \{e \setminus S : e \in E(\mathcal{H}), S \subseteq e \} \). Throughout the rest of the paper, we frequently use the simple identity \( \binom{m}{b} \binom{m-b}{k-b} = \binom{m}{k} \binom{k}{b} \), which holds for all integers \( 0 \leq b \leq k \leq m \).

**Fact 2.9.** Let \( 0 \leq b < k \) and \( m \geq 2k - b \). If \( \mathcal{H} \) is a \( k \)-graph on \( m \) vertices containing no copy of \( \mathcal{Y}_{k,b} \), then \( e(\mathcal{H}) < \binom{m}{k-1} \).

**Proof.** Fix any \( b \)-set \( S \subseteq V(\mathcal{H}) \) (\( S = \emptyset \) if \( b = 0 \)) and consider its link graph \( L_S \). Since \( \mathcal{H} \) contains no copy of \( \mathcal{Y}_{k,b} \), any two edges of \( L_S \) intersect. Since \( m \geq 2k - b \), the Erdős–Ko–Rado Theorem [5] implies that \( |L_S| \leq \binom{m-b-1}{k-b-1} \). Thus,
\[
e(\mathcal{H}) \leq \frac{1}{k} \left( \binom{m}{b} \right) \left( \binom{m-b-1}{k-b-1} \right) = \frac{1}{k} \left( \binom{m}{b} \right) \left( \binom{m-b}{k-b} \right) \frac{k-b}{m-b} = \left( \frac{k}{m} \right) \frac{k-b}{m-b} = \left( \frac{m}{k} \right) \frac{k-b}{m-b}
\]
\[
= \left( \frac{m}{k-1} \right) \frac{k-b}{m-b} < \left( \frac{m}{k-1} \right). \qquad \square
\]

**Proof of Lemma 2.8.** Given \( \gamma, \beta > 0 \), let \( \epsilon' = \frac{\gamma \beta^{k-1}}{(k-1)!} \) and let \( n \in \mathbb{N} \) be sufficiently large. Let \( \mathcal{H} \) be a \( k \)-graph on \( n \) vertices that satisfies \( \deg(S) \geq \left( \frac{1}{2k-b} - \gamma \right) n \) for all but at most
$\epsilon' n^{k-1}$ (k - 1)-sets $S$. Let $\mathcal{Y} = \{Y_1, \ldots, Y_m\}$ be a largest $\mathcal{Y}_{k,b}$-tiling in $\mathcal{H}$ (with respect to $m$) and write $V_i = V(Y_i)$ for $i \in [m]$. Let $V' = \bigcup_{i \in [m]} V_i$ and $U = V(\mathcal{H}) \setminus V'$. Assume that $|U| > \beta n$ — otherwise we are done.

Let $C$ be the set of vertices $v \in V'$ such that $\deg(v, U) \geq (2k - b)^2 \binom{|U|}{k-2}$. We will show that $|C| \leq \frac{n}{2k - b}$ and $C$ covers almost all the edges of $\mathcal{H}$, which implies that $\mathcal{H}[V \setminus C]$ is sparse and $\mathcal{H}$ is in the extremal case. We first observe that every $Y_i \in \mathcal{Y}$ contains at most one vertex in $C$. Suppose instead, two vertices $x, y \in V_i$ are both in $C$. Since $\deg(x, U) \geq (2k - b)^2 \binom{|U|}{k-2} > \binom{|U|}{k-2}$, by Fact 2.9, there is a copy of $\mathcal{Y}_{k-1,b-1}$ in the link graph of $x$ on $U$, which gives rise to $\mathcal{Y}'$, a copy of $\mathcal{Y}_{k,b}$ on $\{x\} \cup U$. Since the link graph of $y$ on $U \setminus V(Y')$ has at least

$$(2k - b)^2 \binom{|U|}{k-2} - (2k - b - 1) \binom{|U|}{k-2} > \binom{|U \setminus V(Y')|}{k-2}$$

edges, we can find another copy of $\mathcal{Y}_{k,b}$ on $\{y\} \cup (U \setminus V(Y'))$ by Fact 2.9. Replacing $Y_i$ in $\mathcal{Y}$ with these two copies of $\mathcal{Y}_{k,b}$ creates a $\mathcal{Y}_{k,b}$-tiling larger than $\mathcal{Y}$, contradiction. Consequently,

$$\sum_{S \in \binom{U}{k-1}} \deg(S, V') \leq |C| \binom{|U|}{k-1} + |V' \setminus C| (2k - b)^2 \binom{|U|}{k-2}$$

$$< |C| \binom{|U|}{k-1} + (2k - b)^2 n \binom{|U|}{k-2} \quad \text{because } |V' \setminus C| < n$$

$$= \left( \binom{|U|}{k-1} |C| + \frac{(2k - b)^2 n (k - 1)}{|U| - k + 2} \right). \quad (2.4)$$

Second, by Fact 2.9, $e(U) \leq \binom{|U|}{k-1}$ since $\mathcal{H}[U]$ contains no copy of $\mathcal{Y}_{k,b}$, which implies

$$\sum_{S \in \binom{U}{k-1}} \deg(S, U) \leq k \binom{|U|}{k-1}. \quad (2.5)$$

By the definition of $\epsilon'$, we have

$$\epsilon' n^{k-1} = \frac{\gamma}{(k-1)!} n^{k-1} < \frac{\gamma |U|^{k-1}}{(k-1)!} < 2\gamma \binom{|U|}{k-1}$$

as $|U|$ is large enough. At last, by the degree condition, we have

$$\sum_{S \in \binom{U}{k-1}} \deg(S) \geq \left( \binom{|U|}{k-1} - \epsilon' n^{k-1} \right) \left( \frac{1}{2k - b} - \gamma \right) n$$

$$> (1 - 2\gamma) \binom{|U|}{k-1} \left( \frac{1}{2k - b} - \gamma \right) n. \quad (2.6)$$
Since \( \text{deg}(S) = \text{deg}(S, U) + \text{deg}(S, V') \), we combine (2.4), (2.5) and (2.6) and get

\[
|C| > (1 - 2\gamma) \left( \frac{1}{2k-b} - \gamma \right) n - k - \frac{(2k-b)^2n(k-1)}{|U| - k + 2}.
\]

Since \( |U| > 16k^3/\gamma \), we get

\[
\frac{(2k-b)^2n(k-1)}{|U| - k + 2} < 4k^3n < \gamma n / 2.
\]

Since \( 2\gamma^2n > k \) and \( 2k-b \geq 4 \), it follows that \( |C| > \left( \frac{1}{2k-b} - 2\gamma \right)n \).

Let \( I_C \) be the set of all \( i \in [m] \) such that \( V_i \cap C \neq \emptyset \). Since each \( V_i, i \in I_C \), contains one vertex of \( C \), we have

\[
|I_C| = |C| \geq \left( \frac{1}{2k-b} - 2\gamma \right)n \geq m - 2\gamma n. \tag{2.7}
\]

Let \( A = (\bigcup_{i \in I_C} V_i \setminus C) \cup U \).

**Claim 2.10.** \( \mathcal{H}[A] \) contains no copy of \( \mathcal{Y}_{k,b} \), thus \( e(A) < \binom{n}{k-1} \).

**Proof.** The first half of the claim implies the second half by Fact 2.9. Suppose instead, \( \mathcal{H}[A] \) contains a copy of \( \mathcal{Y}_{k,b} \), denoted by \( \mathcal{Y}_0 \). Note that \( V(\mathcal{Y}_0) \not\subseteq U \) because \( \mathcal{H}[U] \) contains no copy of \( \mathcal{Y}_{k,b} \). Without loss of generality, suppose that \( V_1, \ldots, V_j \) contain the vertices of \( \mathcal{Y}_0 \) for some \( j \leq 2k-b \). For \( i \in [j] \), let \( c_i \) denote the unique vertex in \( V_i \cap C \). We greedily construct vertex-disjoint copies of \( \mathcal{Y}_{k,b} \) on \( \{c_i\} \cup U, i \in [j] \) as follows. Suppose we have found \( \mathcal{Y}_1', \ldots, \mathcal{Y}_i' \) (copies of \( \mathcal{Y}_{k,b} \)) for some \( i < j \). Let \( U_0 \) denote the set of the vertices of \( U \) covered by \( \mathcal{Y}_0, \mathcal{Y}_1', \ldots, \mathcal{Y}_i' \). Then \( |U_0| \leq (i+1)(2k-b-1) \leq (2k-b)(2k-b-1) \). Since \( \text{deg}(c_{i+1}, U) \geq (2k-b)^2 \binom{|U|}{k-2} \), the link graph of \( c_{i+1} \) on \( U \setminus U_0 \) has at least

\[
(2k-b)^2 \binom{|U|}{k-2} - |U_0| \binom{|U|}{k-2} > \binom{|U|}{k-2}
\]

edges. By Fact 2.9, there is a copy of \( \mathcal{Y}_{k,b} \) on \( \{c_{i+1}\} \cup (U \setminus U_0) \). Let \( \mathcal{Y}_1', \ldots, \mathcal{Y}_j' \) denote the copies of \( \mathcal{Y}_{k,b} \) constructed in this way. Replacing \( \mathcal{Y}_1, \ldots, \mathcal{Y}_j \) in \( \mathcal{Y} \) with \( \mathcal{Y}_0, \mathcal{Y}_1', \ldots, \mathcal{Y}_j' \) gives a \( \mathcal{Y}_{k,b} \)-tiling larger than \( \mathcal{Y} \), contradiction. \( \square \)

Note that the edges not incident to \( C \) are either contained in \( A \) or intersect some \( V_i, i \notin I_C \). By (2.7) and Claim 2.10,

\[
e(V \setminus C) \leq e(A) + (2k-b) \cdot 2\gamma n \binom{n-1}{k-1} < \binom{n}{k-1} + (4k-2b)\gamma n \binom{n}{k-1} \]

\[
< 4k\gamma n \binom{n}{k-1} < \frac{4k}{(k-1)!} \gamma n^k \leq 6\gamma n^k,
\]
where the last inequality follows from $k \geq 3$. Since $|C| \leq \frac{n}{2k-\ell}$, we can pick a set $B \subseteq V \setminus C$ of order \[ \frac{2k-b-1}{2k-b} n \] such that $e(B) < 6\gamma n^k$. \qed

3. The extremal theorem

In this section we prove Theorem 1.5. Assume that $k \geq 3$, $1 \leq \ell < k/2$ and $0 < \Delta \ll 1$. Let $n \in (k - \ell)\mathbb{N}$ be sufficiently large. Let $\mathcal{H}$ be a $k$-graph on $V$ of $n$ vertices such that $\delta_{k-1}(\mathcal{H}) \geq \frac{n}{2(k-\ell)}$. Furthermore, assume that $\mathcal{H}$ is $\Delta$-extremal, namely, there is a set $B \subseteq V(\mathcal{H})$, such that $|B| = \left\lfloor \frac{(2k-2\ell-1)n}{2(k-\ell)} \right\rfloor$ and $e(B) \leq \Delta n^k$. Let $A = V \setminus B$. Then $|A| = \left\lceil \frac{n}{2(k-\ell)} \right\rceil$.

The following is an outline of the proof. We denote by $A'$ and $B'$ the sets of the vertices of $\mathcal{H}$ that behave as typical vertices of $A$ and $B$, respectively. Let $V_0 = V \setminus (A' \cup B')$. It is not hard to show that $A' \approx A$, $B' \approx B$, and thus $V_0 \approx \emptyset$. In the ideal case, when $V_0 = \emptyset$ and $|B'| = (2k - 2\ell - 1)|A'|$, we assign a cyclic order to the vertices of $A'$, construct $|A'|$ copies of $\mathcal{Y}_{k,\ell}$ such that each copy contains one vertex of $A'$ and $2k - \ell - 1$ vertices of $B'$, and any two consecutive copies of $\mathcal{Y}_{k,\ell}$ share exactly $\ell$ vertices of $B'$. This gives rise to the desired Hamilton $\ell$-cycle of $\mathcal{H}$. In the general case, we first construct an $\ell$-path $Q$ with ends $L_0$ and $L_1$ such that $V_0 \subseteq V(Q)$ and $|B_1| = (2k - \ell - 1)|A_1| + \ell$, where $A_1 = A' \setminus V(Q)$ and $B_1 = (B \setminus V(Q)) \cup L_0 \cup L_1$. Next we complete the Hamilton $\ell$-cycle by constructing an $\ell$-path on $A_1 \cup B_1$ with ends $L_0$ and $L_1$.

For the convenience of later calculations, we let $\epsilon_0 = 2k!e\Delta \ll 1$ and claim that $e(B) \leq \epsilon_0 \binom{|B|}{k}$. Indeed, since $2(k-\ell) - 1 \geq k$, we have

$$\frac{1}{e} \leq \left(1 - \frac{1}{2(k-\ell)}\right)^{2(k-\ell)-1} \leq \left(1 - \frac{1}{2(k-\ell)}\right)^k.$$ 

Thus we get

$$e(B) \leq \frac{\epsilon_0}{2k!e} n^k \leq \epsilon_0 \left(1 - \frac{1}{2(k-\ell)}\right)^k \frac{n^k}{2k!} \leq \epsilon_0 \binom{|B|}{k}.$$ 

(3.1)

In general, given two disjoint vertex sets $X$ and $Y$ and two integers $i, j \geq 0$, a set $S \subset X \cup Y$ is called an $X^iY^j$-set if $|S \cap X| = i$ and $|S \cap Y| = j$. When $X, Y$ are two disjoint subsets of $V(\mathcal{H})$ and $i + j = k$, we denote by $\mathcal{H}(X^iY^j)$ the family of all edges of $\mathcal{H}$ that are $X^iY^j$-sets, and let $e_{\mathcal{H}}(X^iY^j) = |\mathcal{H}(X^iY^j)|$ (the subscript may be omitted if it is clear from the context). We use $\bar{e}_{\mathcal{H}}(X^iY^{k-i})$ to denote the number of non-edges among $X^iY^{k-i}$-sets. Given a set $L \subseteq X \cup Y$ with $|L \cap X| = l_1 \leq i$ and $|L \cap Y| = l_2 \leq k - i$, we define $\deg(L, X^iY^{k-i})$ as the number of edges in $\mathcal{H}(X^iY^{k-i})$ that contain $L$, and $\overline{\deg}(L, X^iY^{k-i}) = (|X| - l_1)(|Y| - l_2) - \deg(L, X^iY^{k-i})$. Our earlier notation $\deg(S, R)$ may be viewed as $\deg(S, S^{|S|}(R \setminus S)^{|\bar{S}|})$. 


3.1. Classification of vertices

Let \( \epsilon_1 = \epsilon_0^{1/3} \) and \( \epsilon_2 = 2\epsilon_1^2 \). Assume that the partition \( V(H) = A \cup B \) satisfies that \( |B| = \left\lfloor \frac{(2k-2(\ell-1))n}{2(2k-\ell)} \right\rfloor \) and (3.1). In addition, assume that \( e(B) \) is the smallest among all such partitions. We now define

\[
A' := \left\{ v \in V : \deg(v, B) \geq (1 - \epsilon_1) \binom{|B|}{k-1} \right\},
\]

\[
B' := \left\{ v \in V : \deg(v, B) \leq \epsilon_1 \left( \frac{|B|}{k-1} \right) \right\},
\]

\[
V_0 := V \setminus (A' \cup B').
\]

**Claim 3.1.** \( A \cap B' \neq \emptyset \) implies that \( B \subseteq B' \), and \( B \cap A' \neq \emptyset \) implies that \( A \subseteq A' \).

**Proof.** First, assume that \( A \cap B' \neq \emptyset \). Then there is some \( u \in A \) such that \( \deg(u, B) \leq \epsilon_1 \binom{|B|}{k-1} \). If there exists some \( v \in B \setminus B' \), namely, \( \deg(v, B) > \epsilon_1 \binom{|B|}{k-1} \), then we can switch \( u \) and \( v \) and form a new partition \( A'' \cup B'' \) such that \( |B''| = |B| \) and \( e(B'') < e(B) \), which contradicts the minimality of \( e(B) \).

Second, assume that \( B \cap A' \neq \emptyset \). Then some \( u \in B \) satisfies that \( \deg(u, B) \geq (1 - \epsilon_1) \binom{|B|}{k-1} \). Similarly, by the minimality of \( e(B) \), we get that for any vertex \( v \in A \), \( \deg(v, B) \geq (1 - \epsilon_1) \binom{|B|}{k-1} \), which implies that \( A \subseteq A' \).

**Claim 3.2.** \( \{|A \setminus A'|, |B \setminus B'|, |A' \setminus A|, |B' \setminus B|\} \leq \epsilon_2 B | \) and \( |V_0| \leq 2\epsilon_2 B | \).

**Proof.** First assume that \( |B \setminus B'| > \epsilon_2 B | \). By the definition of \( B' \), we get that

\[
e(B) > \frac{1}{k} \epsilon_1 \binom{|B|}{k-1} \cdot \epsilon_2 |B| > 2\epsilon_0 \binom{|B|}{k-1},
\]

which contradicts (3.1).

Second, assume that \( |A \setminus A'| > \epsilon_2 B | \). Then by the definition of \( A' \), for any vertex \( v \notin A' \), we have that \( \deg(v, B) > \epsilon_1 \binom{|B|}{k-1} \). So we get

\[
e(AB^{k-1}) > \epsilon_2 |B| \cdot \epsilon_1 \binom{|B|}{k-1} = 2\epsilon_0 |B| \binom{|B|}{k-1}.
\]

Together with (3.1), this implies that

\[
\sum_{S \in \binom{B}{k-1}} \overline{\deg(S)} = k e(B) + e(AB^{k-1})
\]

\[
> k (1 - \epsilon_0) \binom{|B|}{k} + 2\epsilon_0 |B| \binom{|B|}{k-1}
\]

\[
= (1 - \epsilon_0) (|B| - k + 1) + 2\epsilon_0 |B| \binom{|B|}{k-1} > |B| \binom{|B|}{k-1}.
\]
where the last inequality holds because \( n \) is large enough. By the pigeonhole principle, there exists a set \( S \subseteq \binom{B}{k-1} \), such that \( \overline{\deg}(S) > |B| = \left\lfloor \frac{(2k-2\ell-1)n}{2(k-\ell)} \right\rfloor \), contradicting (1.1).

Consequently,

\[
|A' \setminus A| = |A' \cap B| \leq |B \setminus B'| \leq \epsilon_2|B|,
\]
\[
|B' \setminus B| = |A \cap B'| \leq |A \setminus A'| \leq \epsilon_2|B|,
\]
\[
|V_0| = |A \setminus A'| + |B \setminus B'| \leq \epsilon_2|B| + \epsilon_2|B| = 2\epsilon_2|B|.
\]

\(\square\)

3.2. Classification of \( \ell \)-sets in \( B' \)

In order to construct our Hamilton \( \ell \)-cycle, we need to connect two \( \ell \)-paths. To make this possible, we want the ends of our \( \ell \)-paths to be \( \ell \)-sets in \( B' \) that have high degree in \( \mathcal{H}[A'B'k-1] \). Formally, we call an \( \ell \)-set \( L \subseteq V \) typical if \( \deg(L, B) \leq \epsilon_1(\binom{|B|}{k-\ell}) \), otherwise atypical. We prove several properties related to typical \( \ell \)-sets in this subsection.

**Claim 3.3.** The number of atypical \( \ell \)-sets in \( B \) is at most \( \epsilon_2(\binom{|B|}{\ell}) \).

**Proof.** Let \( m \) be the number of atypical \( \ell \)-sets in \( B \). By (3.1), we have

\[
\frac{m\epsilon_1(\binom{|B|}{k-\ell})}{\binom{k}{\ell}} \leq e(B) \leq \epsilon_0\left(\binom{|B|}{k}\right),
\]

which gives that \( m \leq \frac{\epsilon_1\binom{k}{\ell}(\binom{|B|}{k})}{\epsilon_1\binom{k}{k-\ell}} = \frac{\epsilon_2}{2}\left(\binom{|B|}{\ell}+k^\ell\right) < \epsilon_2(\binom{|B|}{\ell}). \)

\(\square\)

**Claim 3.4.** Every typical \( \ell \)-set \( L \subseteq B' \) satisfies \( \overline{\deg}(L, A'B'k-1) \leq 4k\epsilon_1(\binom{|B'|}{k-\ell-1})|A'|. \)

**Proof.** Fix a typical \( \ell \)-set \( L \subseteq B' \) and consider the following sum,

\[
\sum_{L \subseteq D \subseteq B', |D|=k-1} \deg(D) = \sum_{L \subseteq D \subseteq B', |D|=k-1} (\deg(D, A') + \deg(D, B') + \deg(D, V_0)).
\]

By (1.1), the left-hand side is at least \( \binom{|B'|}{k-\ell-1}|A'| \). On the other hand,

\[
\sum_{L \subseteq D \subseteq B', |D|=k-1} (\deg(D, B') + \deg(D, V_0)) \leq (k-\ell)\deg(L, B') + \left(\binom{|B'|}{k-\ell-1}\right)|V_0|.
\]

Since \( L \) is typical and \( |B' \setminus B| \leq \epsilon_2|B| \) (Claim 3.2), we have

\[
\deg(L, B') \leq \deg(L, B) + |B' \setminus B| \left(\binom{|B'|}{k-\ell-1}\right)
\leq \epsilon_1\left(\binom{|B|}{k-\ell}\right) + \epsilon_2|B| \left(\binom{|B'|}{k-\ell-1}\right).
\]
Since $\epsilon_2 \ll \epsilon_1$ and $|B| - |B'| \leq \epsilon_2|B|$, it follows that

$$(k - \ell) \deg(L, B') \leq \epsilon_1|B| \left(\frac{|B| - 1}{k - \ell - 1}\right) + (k - \ell)\epsilon_2|B| \left(\frac{|B'| - 1}{k - \ell - 1}\right)$$

$$\leq 2\epsilon_1|B| \left(\frac{|B'| - \ell}{k - \ell - 1}\right).$$

Putting these together and using Claim 3.2, we obtain that

$$\sum_{L \subset D \subset B', |D| = k-1} \deg(D, A') \geq \left(\frac{|B'| - \ell}{k - \ell - 1}\right)(|A| - |V_0|) - 2\epsilon_1|B| \left(\frac{|B'| - \ell}{k - \ell - 1}\right)$$

$$\geq \left(\frac{|B'| - \ell}{k - \ell - 1}\right)(|A'| - 3\epsilon_2|B| - 2\epsilon_1|B|).$$

Note that $\deg(L, A'B'k-1) = \sum_{L \subset D \subset B', |D| = k-1} \deg(D, A')$. Since $|B| \leq (2k - 2\ell - 1)|A| \leq (2k - 2\ell)|A'|$, we finally derive that

$$\deg(L, A'B'k-1) \geq \left(\frac{|B'| - \ell}{k - \ell - 1}\right)(1 - (2k - 2\ell)(3\epsilon_2 + 2\epsilon_1))|A'|$$

$$\geq (1 - 4k\epsilon_1) \left(\frac{|B'| - \ell}{k - \ell - 1}\right)|A'|.$$ as desired. \qed

We next show that we can connect any two disjoint typical $\ell$-sets of $B'$ with an $\ell$-path of length two while avoiding any given set of $\frac{n}{4(k-\ell)}$ vertices of $V$.

Claim 3.5. Given two disjoint typical $\ell$-sets $L_1, L_2$ in $B'$ and a vertex set $U \subseteq V$ with $|U| \leq \frac{n}{4(k-\ell)}$, there exist a vertex $a \in A' \setminus U$ and a $(2k - 3\ell - 1)$-set $C \subset B' \setminus U$ such that $L_1 \cup L_2 \cup \{a\} \cup C$ spans an $\ell$-path (of length two) ended at $L_1, L_2$.

Proof. Fix two disjoint typical $\ell$-sets $L_1, L_2$ in $B'$. Using Claim 3.2, we obtain that $|U| \leq \frac{n}{4(k-\ell)} \leq \frac{|A|}{2} < \frac{2}{3}|A'|$ and

$$\frac{n}{4(k-\ell)} \leq \frac{|B| + 1}{2(2k - 2\ell - 1)} \leq \frac{(1 + 2\epsilon_2)|B'|}{2k} < \frac{|B'|}{k}.$$ Thus $|A' \setminus U| > \frac{|A'|}{3}$ and $|B' \setminus U| > \frac{k-1}{k}|B'|$. Consider a $(k - \ell)$-graph $G$ on $(A' \cup B') \setminus U$ such that an $A'B'k-\ell-1$-set $T$ is an edge of $G$ if and only if $T \cap U = \emptyset$ and $T$ is a common neighbor of $L_1$ and $L_2$ in $\mathcal{H}$. By Claim 3.4, we have
Hence, 

$$e(G) \leq 2 \cdot 4k\epsilon_1 \left(\frac{|B'| - \ell}{k - \ell - 1}\right) |A'| < 8k\epsilon_1 \left(\frac{k}{k-1} \left| B' \setminus U \right| \right) \cdot 3|A' \setminus U|. $$

Consequently, 

$$e(G) > \frac{1}{2} \left(\frac{|B'| \setminus U}{k - \ell - 1}\right) |A' \setminus U|. $$

Hence there exists a vertex $a \in A' \setminus U$ such that $\deg_{G}(a) > \frac{1}{2} \left(\frac{|B'| \setminus U}{k - \ell - 1}\right)$. By Fact 2.9, the link graph of $a$ contains a copy of $\mathcal{Y}_{k-\ell-1,\ell-1}$ (two edges of the link graph sharing $\ell - 1$ vertices). In other words, there exists a $(2k - 3\ell - 1)$-set $C \subset B' \setminus U$ such that $C \cup \{a\}$ contains two edges of $G$ sharing $\ell$ vertices. Together with $L_1, L_2$, this gives rise to the desired $\ell$-path (in $\mathcal{H}$) of length two ended at $L_1, L_2$. \(\square\)

The following claim shows that we can always extend a typical $\ell$-set to an edge of $\mathcal{H}$ by adding one vertex from $A'$ and $k - \ell - 1$ vertices from $B'$ such that every $\ell$-set of these $k - \ell - 1$ vertices is typical. This can be done even when at most $\frac{n}{4(k-\ell)}$ vertices of $V$ are not available.

**Claim 3.6.** Given a typical $\ell$-set $L \subseteq B'$ and a set $U \subseteq V$ with $|U| \leq \frac{n}{4(k-\ell)}$, there exists an $A'B'^{k-\ell-1}$-set $C \subset V \setminus U$ such that $L \cup C$ is an edge of $\mathcal{H}$ and every $\ell$-subset of $C \cap B'$ is typical.

**Proof.** First, since $L$ is typical in $B'$, by Claim 3.4, 

$$\overline{\deg}(L, A'B'^{k-1}) \leq 4k\epsilon_1 \left(\frac{|B'| - \ell}{k - \ell - 1}\right) |A'|. $$

Second, note that a vertex in $A'$ is contained in $(\frac{|B'|}{k-\ell-1})$ $A'B'^{k-\ell-1}$-sets, while a vertex in $B'$ is contained in $|A'| (\frac{|B'| - 1}{k-\ell-2}) A'B'^{k-\ell-1}$-sets. It is easy to see that $|A'| (\frac{|B'| - 1}{k-\ell-2}) < (\frac{|B'|}{k-\ell-1})$ (as $|A'| \approx \frac{n}{2k-2\ell}$ and $|B'| \approx \frac{2k-2\ell-1}{n}$). We thus derive that at most 

$$|U| \left(\frac{|B'|}{k - \ell - 1}\right) \leq \frac{n}{4(k - \ell)} \left(\frac{|B'|}{k - \ell - 1}\right), $$

$A'B'^{k-\ell-1}$-sets intersect $U$. Finally, by Claim 3.3, the number of atypical $\ell$-sets in $B$ is at most $\epsilon_2 (\frac{|B|}{\ell})$. Using Claim 3.2, we derive that the number of atypical $\ell$-sets in $B'$ is at most 

$$\epsilon_2 (\frac{|B|}{\ell}) + |B'| \left(\frac{|B'| - 1}{\ell - 1}\right) \leq 2\epsilon_2 (\frac{|B'|}{\ell}) + \epsilon_2 |B| \left(\frac{|B'| - 1}{\ell - 1}\right) < 3\epsilon_2 (\frac{|B'|}{\ell}). $$

Hence at most $3\ell \epsilon_2 (\frac{|B'|}{\ell}) |A'| (\frac{|B'| - \ell}{k-2\ell-1}) A'B'^{k-\ell-1}$-sets contain an atypical $\ell$-set. In summary, at most

$$4k\epsilon_1 \left(\frac{|B'| - \ell}{k - \ell - 1}\right) |A'| + \frac{n}{4(k - \ell)} \left(\frac{|B'|}{k - \ell - 1}\right) + 3\ell \epsilon_2 \left(\frac{|B'|}{\ell}\right) \left(\frac{|B'| - \ell}{k - 2\ell - 1}\right) |A'|$$

$A'B'^{k-\ell-1}$-sets fail some of the desired properties. Since $\epsilon_1, \epsilon_2 \ll 1$ and $|A'| \approx \frac{n}{2(k-\ell)}$, the desired $A'B'^{k-\ell-1}$-set always exists. \(\square\)
3.3. Building a short path $Q$

First, by the definition of $B$, for any vertex $b \in B'$, we have

$$
\deg(b, B') \leq \deg(b, B) + |B' \setminus B| \left( \frac{|B'| - 1}{k - 2} \right)
$$

$$
\leq \epsilon_1 \left( \frac{|B|}{k - 1} \right) + \epsilon_2 |B| \left( \frac{|B'| - 1}{k - 2} \right) < 2\epsilon_1 \left( \frac{|B|}{k - 1} \right). \tag{3.2}
$$

The following claim is the only place where we used the exact codegree condition (1.1).

**Claim 3.7.** Suppose that $|A \cap B'| = q > 0$. Then there exists a family $P_1$ of $2q$ vertex-disjoint edges in $B'$, each of which contains two disjoint typical $\ell$-sets.

**Proof.** Let $|A \cap B'| = q > 0$. Since $A \cap B' \neq \emptyset$, by Claim 3.1, we have $B \subseteq B'$, and consequently $|B'| = \lfloor \frac{2k - 2\ell - 1}{2(k - \ell)} \rfloor n + q$. By Claim 3.2, we have $q \leq |A \setminus A'| \leq \epsilon_2 |B|$.

Let $B$ denote the family of the edges in $B'$ that contain two disjoint typical $\ell$-sets. We derive a lower bound for $|B|$ as follows. We first pick a $(k - 1)$-subset of $B$ (recall that $B \subseteq B'$) that contains no atypical $\ell$-subset. Since $2\ell \leq k - 1$, such a $(k - 1)$-set contains two disjoint typical $\ell$-sets. By Claim 3.3, there are at most $\epsilon_2 \left( \frac{|B|}{\ell} \right)$ atypical $\ell$-sets in $B \cap B' = B$ and in turn, there are at most $\epsilon_2 \left( \frac{|B|}{\ell} \right) \left( \frac{|B| - \ell}{k - \ell - 1} \right)$ $(k - 1)$-subsets of $B$ that contain an atypical $\ell$-subset. Thus there are at least

$$
\left( \frac{|B|}{k - 1} \right) - \epsilon_2 \left( \frac{|B|}{\ell} \right) \left( \frac{|B| - \ell}{k - \ell - 1} \right) = \left( 1 - \left( \frac{k - 1}{\ell} \right) \epsilon_2 \right) \left( \frac{|B|}{k - 1} \right)
$$

$(k - 1)$-subsets of $B$ that contain no atypical $\ell$-subset. After picking such a $(k - 1)$-set $S \subset B$, we find a neighbor of $S$ by the codegree condition. Since $|B'| = \lfloor \frac{2k - 2\ell - 1}{2(k - \ell)} \rfloor n + q$, by (1.1), we have $\deg(S, B') \geq q$. We thus derive that

$$
|B| \geq \left( 1 - \left( \frac{k - 1}{\ell} \right) \epsilon_2 \right) \left( \frac{|B|}{k - 1} \right) \frac{q}{k},
$$

in which we divide by $k$ because every edge of $B$ is counted at most $k$ times.

We claim that $B$ contains $2q$ disjoint edges. Suppose instead, a maximum matching in $B$ has $i < 2q$ edges. By (3.2), at most $2q k \cdot 2\epsilon_1 \left( \frac{|B|}{k - 1} \right)$ edges of $B'$ intersect the $i$ edges in the matching. Hence, the number of edges of $B$ that are disjoint from these $i$ edges is at least

$$
\frac{q}{k} \left( 1 - \left( \frac{k - 1}{\ell} \right) \epsilon_2 \right) \left( \frac{|B|}{k - 1} \right) - 4k\epsilon_1 q \left( \frac{|B|}{k - 1} \right) \geq \left( \frac{1}{k} - (4k + 1)\epsilon_1 \right) q \left( \frac{|B|}{k - 1} \right) > 0,
$$

as $\epsilon_2 \ll \epsilon_1 \ll 1$. We may thus obtain a matching of size $i + 1$, a contradiction. \qed
Claim 3.8. There exists a non-empty $\ell$-path $Q$ in $\mathcal{H}$ with the following properties:

- $V_0 \subseteq V(Q)$,
- $|V(Q)| \leq 10k\epsilon_2|B|$, 
- the two ends $L_0, L_1$ of $Q$ are typical $\ell$-sets in $B'$, 
- $|B_1| = (2k - 2\ell - 1)|A_1| + \ell$, where $A_1 = A' \setminus V(Q)$ and $B_1 = (B' \setminus V(Q)) \cup L_0 \cup L_1$.

Proof. We split into two cases here.

Case 1. $A \cap B' \neq \emptyset$.

By Claim 3.1, $A \cap B' \neq \emptyset$ implies that $B \subseteq B'$. Let $q = |A \cap B'|$. We first apply Claim 3.7 and find a family $\mathcal{P}_1$ of vertex-disjoint $2q$ edges in $B'$. Next we associate each vertex of $V_0$ with $2k - \ell - 1$ vertices of $B$ (so in $B'$) forming an $\ell$-path of length two such that these $|V_0|$ paths are pairwise vertex-disjoint, and also vertex-disjoint from the paths in $\mathcal{P}_1$, and all these paths have typical ends. To see it, let $V_0 = \{x_1, \ldots, x_{|V_0|}\}$. Suppose that we have found such $\ell$-paths for $x_1, \ldots, x_{i-1}$ with $i \leq |V_0|$. Since $B \subseteq B'$, it follows that $A \setminus A' = (A \cap B') \cup V_0$. Hence $|V_0| + q = |A \setminus A'| \leq \epsilon_2|B|$ by Claim 3.2. Therefore

$$(2k - \ell - 1)(i - 1) + |V(\mathcal{P}_1)| < 2k|V_0| + 2kq \leq 2k\epsilon_2|B|$$

and consequently at most $2k\epsilon_2|B|\binom{|B|-1}{k-2} < 2k^2\epsilon_2\binom{|B|}{k-1} (k - 1)$-sets of $B$ intersect the existing paths (including $\mathcal{P}_1$). By the definition of $V_0$, $\deg(x_i, B) > \epsilon_1 k^{-1}$. Let $G_{x_i}$ be the $(k - 1)$-graph on $B$ such that $e \in G_{x_i}$ if

- $\{x_i\} \cup e \in E(\mathcal{H})$,
- $e$ does not contain any vertex from the existing paths,
- $e$ does not contain any atypical $\ell$-set.

By Claim 3.3, the number of $(k - 1)$-sets in $B$ containing at least one atypical $\ell$-set is at most $\epsilon_2\binom{|B|}{k-1} \binom{k-\ell-1}{k-1} = \epsilon_2\binom{|B|}{k-1} (k - 1)$. Thus, we have

$$e(G_{x_i}) \geq \epsilon_1 \binom{|B|}{k-1} - 2k^2\epsilon_2 \binom{|B|}{k-1} - \epsilon_2 k^{-1} \binom{|B|}{k-1} > \frac{\epsilon_1}{2} k^{-1} \binom{|B|}{k-1} > \binom{|B|}{k-2},$$

because $\epsilon_2 \ll \epsilon_1$ and $|B|$ is sufficiently large. By Fact 2.9, $G_{x_i}$ contains a copy of $\mathcal{Y}_{k-1,\ell-1}$, which gives the desired $\ell$-path of length two containing $x_i$.

Denote by $\mathcal{P}_2$ the family of $\ell$-paths we obtained so far. Now we need to connect paths of $\mathcal{P}_2$ together to a single $\ell$-path. For this purpose, we apply Claim 3.5 repeatedly to connect the ends of two $\ell$-paths while avoiding previously used vertices. This is possible because $|V(\mathcal{P}_2)| = (2k - \ell)|V_0| + 2kq$ and $(2k - 3\ell)(|V_0| + 2q - 1)$ vertices are needed to connect all the paths in $\mathcal{P}_2$ — the set $U$ (when we apply Claim 3.5) thus satisfies

$$|U| \leq (4k - 4\ell)|V_0| + (6k - 6\ell)q - 2k + 3\ell \leq 6(k - \ell)\epsilon_2|B| - 2k + 3\ell.$$
Let \( \mathcal{P} \) denote the resulting \( \ell \)-path. We have 
\[
|V(\mathcal{P}) \cap A'| = |V_0| + 2q - 1 \text{ and }
\]
\[
|V(\mathcal{P}) \cap B'| = k \cdot 2q + (2k - \ell - 1)|V_0| + (2k - 3\ell - 1)(|V_0| + 2q - 1)
= 2(2k - 2\ell - 1)|V_0| + 2(3k - 3\ell - 1)q - (2k - 3\ell - 1).
\]

Let \( s = (2k - 2\ell - 1)|A' \setminus V(\mathcal{P})| - |B' \setminus V(\mathcal{P})| \). We have 
\[
s = (2k - 2\ell - 1)(|A'| - |V_0| - 2q + 1) - |B'| + 2(2k - 2\ell - 1)|V_0|
+ 2(3k - 3\ell - 1)q - (2k - 3\ell - 1)
= (2k - 2\ell - 1)|A'| - |B'| + (2k - 2\ell - 1)|V_0| + (2k - 2\ell)q + \ell.
\]

Since \( |A'| + |B'| + |V_0| = n \), we have
\[
s = (2k - 2\ell)(|A'| + |V_0| + q) - n + \ell. \tag{3.3}
\]

Note that \( |A'| + |V_0| + q = |A| \) and
\[
(2k - 2\ell)|A| - n = \begin{cases} 0, & \text{if } \frac{n}{k-\ell} \text{ is even,} \\ k - \ell, & \text{if } \frac{n}{k-\ell} \text{ is odd.} \end{cases} \tag{3.4}
\]

Thus \( s = \ell \) or \( s = k \). If \( s = k \), then we extend \( \mathcal{P} \) to an \( \ell \)-path \( \mathcal{Q} \) by applying Claim 3.6, otherwise let \( \mathcal{Q} = \mathcal{P} \). Then
\[
|V(\mathcal{Q})| \leq |V(\mathcal{P})| + (k - \ell) \leq 6k\epsilon_2|B|,
\]
and \( \mathcal{Q} \) has two typical ends \( L_0, L_1 \subset B' \). We claim that
\[
(2k - 2\ell - 1)|A' \setminus V(\mathcal{Q})| - |B' \setminus V(\mathcal{Q})| = \ell. \tag{3.5}
\]

Indeed, when \( s = \ell \), this is obvious; when \( s = k \), \( V(\mathcal{Q}) \setminus V(\mathcal{P}) \) contains one vertex of \( A' \) and \( k - \ell - 1 \) vertices of \( B' \) and thus
\[
(2k - 2\ell - 1)|A' \setminus V(\mathcal{Q})| - |B' \setminus V(\mathcal{Q})| = s - (2k - 2\ell - 1) + (k - \ell - 1) = \ell.
\]

Let \( A_1 = A' \setminus V(\mathcal{Q}) \) and \( B_1 = (B' \setminus V(\mathcal{Q})) \cup L_0 \cup L_1 \). We derive that \( |B_1| = (2k - 2\ell - 1)|A_1| + \ell \) from (3.5).

Case 2. \( A \cap B' = \emptyset \).

Note that \( A \cap B' = \emptyset \) means that \( B' \subseteq B \). Then we have
\[
|A'| + |V_0| = |V \setminus B'| = |A| + |B \setminus B'|. \tag{3.6}
\]

If \( V_0 \neq \emptyset \), we handle this case similarly as in Case 1 except that we do not need to construct \( \mathcal{P}_1 \). By Claim 3.2, \( |B \setminus B'| \leq \epsilon_2|B| \) and thus for any vertex \( x \in V_0 \),
\[
\deg(x, B') \geq \deg(x, B) - |B \setminus B'| \cdot \left(\frac{|B| - 1}{k - 2}\right) \\
\geq \epsilon_1 \left(\frac{|B|}{k - 1}\right) - (k - 1)\epsilon_2 \left(\frac{|B|}{k - 1}\right) > \frac{\epsilon_1}{2} \left(\frac{|B'|}{k - 1}\right). \quad (3.7)
\]

As in Case 1, we let \( V_0 = \{x_1, \ldots, x_{|V_0|}\} \) and cover them with vertex-disjoint \( \ell \)-paths of length two. Indeed, for each \( i \leq |V_0| \), we construct \( G_x \) as before and show that \( e(G_{x_i}) \geq \frac{\omega_1}{4} (|B'|) \). We then apply Fact 2.9 to \( G_{x_i} \), obtaining a copy of \( \mathcal{Y}_{k-1, \ell-1} \), which gives an \( \ell \)-path of length two containing \( x_i \). As in Case 1, we connect these paths to a single \( \ell \)-path \( P \) by applying Claim 3.5 repeatedly. Then \( |V(P)| = (2k-\ell)|V_0|+(2k-3\ell)(|V_0|-1) \). Define \( s \) as in Case 1. Thus (3.3) holds with \( q = 0 \). Applying (3.6) and (3.4), we derive that

\[
s = 2(k-\ell)(|A| + |B' \setminus B'|) - n + \ell \\
= \left\{ \begin{array}{ll} \\
\ell + 2(k-\ell)|B \setminus B'|, & \text{if } \frac{n}{k-\ell} \text{ is even,} \\
k + 2(k-\ell)|B \setminus B'|, & \text{if } \frac{n}{k-\ell} \text{ is odd,}
\end{array} \right. \quad (3.8)
\]

which implies that \( s \equiv \ell \mod (k-\ell) \). We extend \( P \) to an \( \ell \)-path \( Q \) by applying Claim 3.6 \( \frac{s-\ell}{k-\ell} \) times. Then

\[
|V(Q)| = |V(P)| + s - \ell \leq (4k-4\ell)|V_0| - 2k + 3\ell + k - \ell + 2(k-\ell)|B \setminus B'| \\
\leq 10k\epsilon_2|B|
\]

by Claim 3.2. Note that \( Q \) has two typical ends \( L_0, L_1 \subset B' \). Since \( V(Q) \setminus V(P) \) contains \( \frac{s-\ell}{k-\ell} \) vertices of \( A' \) and \( \frac{s-\ell}{k-\ell} (k-\ell-1) \) vertices of \( B' \), we have

\[
(2k - 2\ell - 1)|A' \setminus V(Q)| - |B' \setminus V(Q)| \\
= s - \frac{s-\ell}{k-\ell} (2k - 2\ell - 1) + \frac{s-\ell}{k-\ell} (k - \ell - 1) = \ell.
\]

We define \( A_1 \) and \( B_1 \) in the same way and similarly we have \(|B_1| = (2k-2\ell-1)|A_1| + \ell \).

When \( V_0 = \emptyset \), we pick an arbitrary vertex \( v \in A' \) and form an \( \ell \)-path \( P \) of length two with typical ends such that \( v \) is in the intersection of the two edges. This is possible by the definition of \( A' \). Define \( s \) as in Case 1. It is easy to see that (3.8) still holds. We then extend \( P \) to \( Q \) by applying Claim 3.6 \( \frac{s-\ell}{k-\ell} \) times. Then \(|V(Q)| = 2k-\ell + s - \ell \leq 2k\epsilon_2|B| \) because of (3.8). The rest is the same as in the previous case. \( \square \)

**Claim 3.9.** The \( A_1, B_1 \) and \( L_0, L_1 \) defined in Claim 3.8 satisfy the following properties:

1. \(|B_1| \geq (1 - \epsilon_1)|B|\),
2. for any vertex \( v \in A_1 \), \( \overline{\deg(v, B_1)} < 3\epsilon_1 (|B_1|) \),
3. for any vertex \( v \in B_1 \), \( \overline{\deg(v, A_1 B_1^{k-1})} \leq 3k\epsilon_1 (|B_1|) \),
4. \( \overline{\deg(L_0, A_1 B_1^{k-1})} \leq 5k\epsilon_1 (|B_1|) \), \( \overline{\deg(L_1, A_1 B_1^{k-1})} \leq 5k\epsilon_1 (|B_1|) \).
**Proof.** Part (1): By Claim 3.2, we have $|B_1 \setminus B| \leq |B' \setminus B| \leq \epsilon_2 |B|$. Furthermore, 
\[ |B_1| \geq |B'| - |V(Q)| \geq |B| - \epsilon_2 |B| - 10k\epsilon_2 |B| \geq (1 - \epsilon_1)|B|. \]
Part (2): For a vertex $v \in A_1$, since $\overline{\deg}(v, B) \leq \epsilon_1 \left( |B|_{k-1} \right)$, we have 
\[ \overline{\deg}(v, B_1) \leq \overline{\deg}(v, B) + |B_1 \setminus B| \left( \frac{|B_1| - 1}{k - 2} \right) \]
\[ \leq \epsilon_1 \left( \frac{|B|}{k - 1} \right) + \epsilon_2 |B| \left( \frac{|B_1| - 1}{k - 2} \right) \]
\[ < \epsilon_1 \left( \frac{|B|}{k - 1} \right) + \epsilon_1 \left( \frac{|B_1|}{k - 1} \right) \leq 3\epsilon_1 \left( \frac{|B_1|}{k - 1} \right), \]
where the last inequality follows from Part (1).
Part (3): Consider the sum $\sum \deg(S \cup \{v\})$ taken over all $S \in \binom{B \setminus \{v\}}{k-2}$. Since $\delta_{k-1}(H) \geq |A|$, we have $\sum \deg(S \cup \{v\}) \geq \binom{|B'|-1}{k-2} |A|$. On the other hand, 
\[ \sum \deg(S \cup \{v\}) = \deg(v, A'B'^{k-1}) + \deg(v, V_0B'^{k-1}) + (k - 1) \deg(v, B'). \]

We thus derive that 
\[ \deg(v, A'B'^{k-1}) \geq \left( \frac{|B'| - 1}{k - 2} \right) |A| - \deg(v, V_0B'^{k-1}) - (k - 1) \deg(v, B'). \]

By Claim 3.2 and (3.2), it follows that 
\[ \deg(v, A'B'^{k-1}) \geq \left( \frac{|B'| - 1}{k - 2} \right) \left( |A'| - \epsilon_2 |B| \right) - 2\epsilon_2 |B| \left( \frac{|B'| - 1}{k - 2} \right) - 2(k - 1)\epsilon_1 \left( \frac{|B|}{k - 1} \right) \]
\[ \geq \left( \frac{|B'| - 1}{k - 2} \right) |A'| - 2k\epsilon_1 \left( \frac{|B|}{k - 1} \right). \]

By Part (1), we now have 
\[ \overline{\deg}(v, A_1B_1^{k-1}) \leq \overline{\deg}(v, A'B'^{k-1}) \leq 2k\epsilon_1 \left( \frac{|B|}{k - 1} \right) \leq 3k\epsilon_1 \left( \frac{|B_1|}{k - 1} \right). \]
Part (4): By Claim 3.4, for any typical $L \subseteq B'$, we have $\overline{\deg}(L, A'B'^{k-1}) \leq 4k\epsilon_1 \left( \frac{|B'|-\ell}{k-\ell-1} \right) |A'|$. Thus, 
\[ \overline{\deg}(L_0, A_1B_1^{k-1}) \leq \overline{\deg}(L_0, A'B'^{k-1}) \leq 4k\epsilon_1 \left( \frac{|B'| - \ell}{k - \ell - 1} \right) |A'| \leq 5k\epsilon_1 \left( \frac{|B_1|}{k - \ell} \right), \]
where the last inequality holds because $|B'| \leq |B_1| + |V(Q)| \leq (1 + \epsilon_1)|B_1|$. The same holds for $L_1$. □
3.4. Completing the Hamilton cycle

We finally complete the proof of Theorem 1.5 by applying the following lemma with \( X = A_1, Y = B_1, \rho = 5k\varepsilon_1, \) and \( L_0, L_1. \)

Lemma 3.10. Fix \( 1 \leq \ell \leq k/2. \) Let \( 0 < \rho \ll 1 \) and \( n \) be sufficiently large. Suppose that \( \mathcal{H} \) is a \( k \)-graph with a partition \( V(\mathcal{H}) = X \cup Y \) and the following properties:

- \( |Y| = (2k - 2\ell - 1)|X| + \ell, \)
- for every vertex \( v \in X, \deg(v, Y) \leq \rho|Y| \) and for every vertex \( v \in Y, \deg(v, XY^{k-1}) \leq \rho|Y|, \)
- there are two disjoint \( \ell \)-sets \( L_0, L_1 \subset Y \) such that
  \[
  \deg(L_0, XY^{k-1}), \deg(L_1, XY^{k-1}) \leq \rho\left(\frac{|Y|}{k - \ell}\right).
  \]

Then \( \mathcal{H} \) contains a Hamilton \( \ell \)-path with \( L_0 \) and \( L_1 \) as ends.

In order to prove Lemma 3.10, we apply two results of Glebov, Person, and Weps [6]. Given \( 1 \leq j \leq k - 1 \) and \( 0 \leq \rho \leq 1, \) an ordered set \( (x_1, \ldots, x_j) \) is \( \rho \)-typical in a \( k \)-graph \( \mathcal{G} \) if for every \( i \in [j], \)

\[
\overline{\deg}_\mathcal{G}(\{x_1, \ldots, x_i\}) \leq \rho^{k-i}\left(\frac{|V(\mathcal{G})| - i}{k - i}\right).
\]

It was shown in [6] that every \( k \)-graph \( \mathcal{G} \) with very large minimum vertex degree contains a tight Hamilton cycle. The proof of [6, Theorem 2] actually shows that we can obtain a tight Hamilton cycle by extending any fixed tight path of constant length with two typical ends. This implies the following theorem that we will use.

Theorem 3.11. (See [6].) Given \( 1 \leq j \leq k \) and \( 0 < \alpha < 1, \) there exists an \( m_0 \) such that the following holds. Suppose that \( \mathcal{G} \) is a \( k \)-graph on \( V \) with \( |V| = m \geq m_0 \) and \( \delta_1(\mathcal{G}) \geq (1 - \alpha)(\frac{m-1}{k-1}). \) Then given any two disjoint \( (22\alpha)^{\frac{1}{k-1}} \)-typical ordered \( j \)-sets \( (x_1, \ldots, x_j) \) and \( (y_1, \ldots, y_j), \) there exists a tight Hamilton path \( P = x_jx_{j-1}\cdots x_1y_1y_2\cdots y_j \) in \( \mathcal{G}. \)

We also use [6, Lemma 3], in which \( V^{2k-2} \) denotes the set of all \((2k-2)\)-tuples \((v_1, \ldots, v_{2k-2})\) such that \( v_i \in V \) (\( v_i \)'s are not necessarily distinct).

Lemma 3.12. (See [6].) Let \( \mathcal{G} \) be the \( k \)-graph given in Lemma 3.11. Suppose that \((x_1, \ldots, x_{2k-2})\) is selected uniformly at random from \( V^{2k-2}. \) Then the probability that all \( x_i \)'s are pairwise distinct and \((x_1, \ldots, x_{k-1}), (x_k, \ldots, x_{2k-2})\) are \((22\alpha)^{\frac{1}{k-1}}\)-typical is at least \( \frac{8}{11}. \)
Proof of Lemma 3.10. In this proof we often write the union $A \cup B \cup \{x\}$ as $ABx$, where $A,B$ are sets and $x$ is an element.

Let $t = |X|$. Our goal is to write $X$ as $\{x_1, \ldots, x_t\}$ and partition $Y$ as $\{L_i, R_i, S_i, R'_i : i \in [t]\}$ with $|L_i| = \ell$, $|R_i| = |R'_i| = k - 2\ell$, and $|S_i| = \ell - 1$ such that

$$L_iR_iS_ix_i, S_ix_iR'_iL_{i+1} \in E(H)$$  (3.10)

for all $i \in [t]$, where $L_{t+1} = L_0$. Consequently,

$$L_1R_1S_1x_1R'_1L_2R_2S_2x_2R'_2 \cdots L_tR_tS_tx_tR'_tL_{t+1}$$

is the desired Hamilton $\ell$-path of $H$.

Let $G$ be the $(k-1)$-graph on $Y$ whose edges are all $(k-1)$-sets $S \subseteq Y$ such that $\deg_H(S,X) > (1 - \sqrt{\rho})t$. The following is an outline of our proof. We first find a small subset $Y_0 \subseteq Y$ with a partition $\{L_i, R_i, S_i, R'_i : i \in [t_0]\}$ such that for every $x \in X$, we have $L_iR_iS_ix_iR'_iL_{i+1} \in E(H)$ for many $i \in [t_0]$. Next we apply Theorem 3.11 to $G[Y \setminus Y_0]$ and obtain a tight Hamilton path, which, in particular, partitions $Y \setminus Y_0$ into $\{L_i, R_i, S_i, R'_i : t_0 < i \leq t\}$ such that $L_iR_iS_iR'_iL_{i+1} \in E(G)$ for $t_0 < i \leq t$. Finally we apply the Marriage Theorem to find a perfect matching between $X$ and $[t]$ such that (3.10) holds for all matched $x_i$ and $i$.

We now give details of the proof. First we claim that

$$\delta_1(G) \geq (1 - 2\sqrt{\rho})\binom{|Y| - 1}{k - 2};$$  (3.11)

and consequently,

$$\varepsilon(G) \leq 2\sqrt{\rho}\binom{|Y|}{k - 1}.  \quad (3.12)$$

Suppose instead, some vertex $v \in Y$ satisfies $\overline{\deg_G(v)} > 2\sqrt{\rho}\binom{|Y| - 1}{k - 2}$. Since every non-neighbor $S'$ of $v$ in $G$ satisfies $\overline{\deg_H(S'v,X)} \geq \sqrt{\rho}t$, we have $\overline{\deg_H(v,XY^{k-1})} > 2\sqrt{\rho}\binom{|Y| - 1}{k - 2} \sqrt{\rho}t$. Since $|Y| = (2k - 2\ell - 1)t + \ell$, we have

$$\overline{\deg_H(v,XY^{k-1})} > 2\rho \frac{|Y| - \ell}{2k - 2\ell - 1} \frac{|Y| - 1}{k - 2} > \rho \frac{|Y|}{k - 1} \frac{|Y| - 1}{k - 2} = \rho \frac{|Y|}{k - 1},$$

contradicting our assumption (the second inequality holds because $|Y|$ is sufficiently large).

Let $Q$ be a $(2k-\ell -1)$-subset of $Y$. We call $Q$ good (otherwise bad) if every $(k-1)$-subset of $Q$ is an edge of $G$ and every $\ell$-set $L \subset Q$ satisfies

$$\overline{\deg_G(L)} \leq \rho^{1/4} \binom{|Y| - \ell}{k - \ell - 1}.  \quad (3.13)$$
Furthermore, we say $Q$ is suitable for a vertex $x \in X$ if $x \cup T \in E(H)$ for every $(k-1)$-set $T \subset Q$. Note that if a $(2k - \ell - 1)$-set is good, by the definition of $G$, it is suitable for at least $(1 - (\frac{2k-\ell-1}{k-1})\sqrt{\rho})t$ vertices of $X$. Let $Y' = Y \setminus (L_0 \cup L_1)$.

**Claim 3.13.** For any $x \in X$, at least $(1 - \rho^{1/3})(\frac{|x|}{2k-\ell-1}) (2k - \ell - 1)$-subsets of $Y'$ are good and suitable for $x$.

**Proof.** Since $\rho + \rho^{1/2} + 3(\frac{2k-\ell-1}{\ell})\rho^{1/4} \leq \rho^{1/3}$, the claim follows from the following three assertions:

- At most $2\ell(\frac{|Y|}{2k-\ell-2}) \leq \rho(\frac{|Y|}{2k-\ell-1}) (2k - \ell - 1)$-subsets of $Y$ are not subsets of $Y'$.
- Given $x \in X$, at most $\rho^{1/2}(\frac{|Y|}{2k-\ell-1}) (2k - \ell - 1)$-sets in $Y$ are not suitable for $x$.
- At most $3(\frac{2k-\ell-1}{\ell})\rho^{1/4}(\frac{|Y|}{2k-\ell-1}) (2k - \ell - 1)$-sets in $Y$ are bad.

The first assertion holds because $|Y \setminus Y'| = 2\ell$. The second assertion follows from the degree condition of $H$, namely, for any $x \in X$, the number of $(2k - \ell - 1)$-sets in $Y$ that are not suitable for $x$ is at most $\rho(\frac{|Y|}{2k-\ell-1}) (\frac{|Y|}{k-\ell} - \frac{k+1}{k-\ell}) \leq \sqrt{\rho}(\frac{|Y|}{2k-\ell-1})$.

To see the third one, let $m$ be the number of $\ell$-sets $L \subseteq Y$ that fail (3.13). By (3.12),

$$m\frac{\rho^{1/4}(\frac{|Y|}{k-\ell-1})}{(\frac{k-\ell-1}{\ell})} \leq \tilde{e}(G) \leq 2\sqrt{\rho}(\frac{|Y|}{k-1}),$$

which implies that $m \leq 2\rho^{1/4}(\frac{|Y|}{k-\ell})$. Thus at most

$$2\rho^{1/4}(\frac{|Y|}{k-\ell-1}) \cdot (\frac{|Y| - \ell}{2k - 2\ell - 1})$$

$(2k - \ell - 1)$-subsets of $Y$ contain an $\ell$-set $L$ that fails (3.13). On the other hand, by (3.12), at most

$$\tilde{e}(G)(\frac{|Y| - k + 1}{k-\ell}) \leq 2\sqrt{\rho}(\frac{|Y|}{k-1}) (\frac{|Y| - k + 1}{k-\ell})$$

$(2k - \ell - 1)$-subsets of $Y$ contain a non-edge of $G$. Putting these together, the number of bad $(2k - \ell - 1)$-sets in $Y$ is at most

$$2\rho^{1/4}(\frac{|Y|}{\ell})(\frac{|Y| - \ell}{2k - 2\ell - 1}) + 2\sqrt{\rho}(\frac{|Y|}{k-1}) (\frac{|Y| - k + 1}{k-\ell})$$

$$\leq 3(\frac{2k - \ell - 1}{\ell})\rho^{1/4}(\frac{|Y|}{2k - \ell - 1}),$$

as $\rho \ll 1$. □
Let $\mathcal{F}_0$ be the set of good $(2k - \ell - 1)$-sets in $Y'$. We will pick a family of disjoint good $(2k - \ell - 1)$-sets in $Y'$ such that for any $x \in X$, many members of this family are suitable for $x$. To achieve this, we pick a family $\mathcal{F}$ by selecting each member of $\mathcal{F}_0$ randomly and independently with probability $p = 6\sqrt{\rho}|Y|/(2k - \ell - 1)$. Then $|\mathcal{F}|$ follows the binomial distribution $B(|\mathcal{F}_0|, p)$ with expectation $E(|\mathcal{F}|) = p|\mathcal{F}_0| \leq p(2k - \ell - 1)$. Furthermore, for every $x \in X$, let $f(x)$ denote the number of members of $\mathcal{F}$ that are suitable for $x$. Then $f(x)$ follows the binomial distribution $B(N, p)$ with $N \geq (1 - \rho^{1/5})(2k - \ell - 1)$ by Claim 3.13. Hence $E(f(x)) \geq p(1 - \rho^{1/5})(2k - \ell - 1)$. Since there are at most $(2k - \ell - 1) \cdot (2k - \ell - 2)$ pairs of intersecting $(2k - \ell - 1)$-sets in $Y$, the expected number of intersecting pairs of $(2k - \ell - 1)$-sets in $\mathcal{F}$ is at most

$$p^2 \left( \frac{|Y|}{2k - \ell - 1} \right) \cdot (2k - \ell - 1) \cdot \left( \frac{|Y| - 1}{2k - \ell - 2} \right) = 36(2k - \ell - 1)^2 \rho|Y|.$$

By Chernoff’s bound (the first two properties) and Markov’s bound (the last one), we can find a family $\mathcal{F}$ of good $(2k - \ell - 1)$-subsets of $Y'$ that satisfies

- $|\mathcal{F}| \leq 2p(2k - \ell - 1) \leq 12\sqrt{\rho}|Y|$, 
- for any vertex $x \in X$, at least $\frac{p}{2}(1 - \rho^{1/5})(2k - \ell - 1) \geq 2\sqrt{\rho}|Y|$ members of $\mathcal{F}$ are suitable for $x$. 
- the number of intersecting pairs of $(2k - \ell - 1)$-sets in $\mathcal{F}$ is at most $72(2k - \ell - 1)^2 \rho|Y|$. 

After deleting one $(2k - \ell - 1)$-set from each of the intersecting pairs from $\mathcal{F}$, we obtain a family $\mathcal{F}' \subseteq \mathcal{F}$ consisting of at most $12\sqrt{\rho}|Y|$ disjoint good $(2k - \ell - 1)$-subsets of $Y'$ and for each $x \in X$, at least

$$2\sqrt{\rho}|Y| - 72(2k - \ell - 1)^2 \rho|Y| \geq \frac{3}{2} \sqrt{\rho}|Y| \quad (3.14)$$

members of $\mathcal{F}'$ are suitable for $x$.

Denote $\mathcal{F}'$ by $\{Q_2, Q_4, \ldots, Q_{2q}\}$ for some $q \leq 12\sqrt{\rho}|Y|$. We arbitrarily partition each $Q_{2i}$ into $L_{2i} \cup P_{2i} \cup L_{2i+1}$ such that $|L_{2i}| = |L_{2i+1}| = \ell$ and $|P_{2i}| = 2k - 3\ell - 1$. Since $Q_{2i}$ is good, both $L_{2i}$ and $L_{2i+1}$ satisfy (3.13). We claim that $L_0$ and $L_1$ satisfy (3.13) as well. Let us show this for $L_0$. By the definition of $\mathcal{G}$, the number of $XY^{k-\ell-1}$-sets $T$ such that $T \cup L_0 \notin E(H)$ is at least $\deg_G(L_0)/\sqrt{pt}$. Using (3.9), we derive that $\deg_G(L_0)/\sqrt{pt} \leq \rho_k(|Y|)$. Since $|Y| \leq (2k - 2\ell)t$, it follows that $\deg_G(L_0) \leq 2\sqrt{\rho}(2k - \ell - 1) \leq \rho^{1/4}(|Y|)$. 

Next we greedily find disjoint $(2k - 3\ell - 1)$-sets $P_1, P_3, \ldots, P_{2q-1}$ from $Y' \setminus \bigcup_{i=1}^{q} Q_{2i}$ such that for each $i \in [q]$, every $(k - \ell - 1)$-subset of $P_{2i-1}$ is a common neighbor of $L_{2i-1}$ and $L_{2i}$ in $\mathcal{G}$. Suppose that we have found $P_1, P_3, \ldots, P_{2i-1}$ for some $i < q$. Since both $L_{2i-1}$ and $L_{2i}$ satisfy (3.13), at most

$$2 \cdot \rho^{1/4} \left( \frac{|Y| - \ell}{k - \ell - 1} \right) \left( \frac{|Y| - k + 1}{k - 2\ell} \right)$$
(2k − 3ℓ − 1)-subsets of Y contain a non-neighbor of L_{2i−1} or L_{2i}. Thus, the number of (2k − 3ℓ − 1)-sets that can be chosen as P_{2i+1} is at least
\[
\left(\frac{|Y'| - (2k - 2\ell - 1)2q}{2k - 3\ell - 1}\right) - 2\cdot \rho^{1/4} \left(\frac{|Y| - \ell}{k - \ell - 1}\right) \left(\frac{|Y| - k + 1}{k - 2\ell}\right) > 0,
\]
as \( q \leq 12\sqrt{\rho|Y|} \) and \( \rho \ll 1 \).
Let \( Y_1 = Y' \setminus \bigcup_{i=1}^{q}(P_{2i-1} \cup Q_{2i}) \) and \( G' = G[Y_1] \). Then \( |Y_1| = |Y'| - (2k - 2\ell - 1)2q \).
Since \( \deg_{G'}(v) \leq \deg_{G}(v) \) for every \( v \in Y_1 \), we have, by (3.11),
\[
\delta_1(G') \geq \left(\frac{|Y_1| - 1}{k - 2}\right) - 2\sqrt{\rho} \left(\frac{|Y| - 1}{k - 2}\right) \geq (1 - 3\sqrt{\rho}) \left(\frac{|Y_1| - 1}{k - 2}\right).
\]
Let \( \alpha = 3\sqrt{\rho} \) and \( \rho_0 = (22\alpha)^{\frac{1}{2}} \). We want to find two disjoint \( \rho_0 \)-typical ordered \((k - \ell - 1)\)-subsets \((x_1, \ldots, x_{k-\ell-1}) \) and \((y_1, \ldots, y_{k-\ell-1}) \) of \( Y_1 \) such that
\[
L_{2q+1} \cup \{x_1, \ldots, x_{k-\ell-1}\}, L_0 \cup \{y_1, \ldots, y_{k-\ell-1}\} \in E(G). \tag{3.15}
\]
To achieve this, we choose \((x_1, \ldots, x_{k-1}, y_1, \ldots, y_{k-1}) \) from \( Y_1^{2k-2} \) uniformly at random. By Lemma 3.12, with probability at least \( \frac{8}{\ell^2}, (x_1, \ldots, x_{k-\ell-1}) \) and \((y_1, \ldots, y_{k-\ell-1}) \) are two disjoint ordered \( \rho_0 \)-typical \((k - \ell - 1)\)-sets. Since \( L_0 \) satisfies (3.13), at most \((k - \ell - 1)!\rho^{1/4}(\frac{|Y| - \ell}{k - \ell - 1}) \) ordered \((k - \ell - 1)\)-subsets of \( Y \) are not neighbors of \( L_0 \) (the same holds for \( L_{2q+1} \)). Thus (3.15) fails with probability at most \((2k - \ell - 1)!\rho^{1/4} \), provided that \( x_1, \ldots, x_{k-\ell-1}, y_1, \ldots, y_{k-\ell-1} \) are all distinct. Therefore the desired \((x_1, \ldots, x_{k-\ell-1}) \) and \((y_1, \ldots, y_{k-\ell-1}) \) exist.
Next we apply Theorem 3.11 to \( G' \) and obtain a tight Hamilton path
\[
P = x_{k-\ell-1}x_{k-\ell-2} \cdots x_1 \cdots y_1y_2 \cdots y_{k-\ell-1}.
\]
Following the order of \( P \), we partition \( Y_1 \) into
\[
R_{2q+1}, S_{2q+1}, R'_{2q+1}, L_{2q+2}, \ldots, L_t, R_t, S_t, R'_t
\]
such that \( |L_i| = \ell, |R_i| = |R'_i| = k - 2\ell, \) and \( |S_i| = \ell - 1. \) Since \( P \) is a tight path in \( G \), we have
\[
L_i R_i S_i, S_i R'_i L_{i+1} \in E(G) \tag{3.16}
\]
for \( 2q + 2 \leq i \leq t - 1 \). Letting \( L_{t+1} = L_0 \), by (3.15), we also have (3.16) for \( i = 2q + 1 \) and \( i = t \).
We now arbitrarily partition \( P_i \), \( 1 \leq i \leq 2q \) into \( R_i \cup S_i \cup R'_i \) such that \( |R_i| = |R'_i| = k - 2\ell, \) and \( |S_i| = \ell - 1. \) By the choice of \( P_i \), (3.16) holds for \( 1 \leq i \leq 2q \).
Consider the bipartite graph \( \Gamma' \) between \( X \) and \( Z := \{z_1, z_2, \ldots, z_t\} \) such that \( x \in X \) and \( z_i \in Z \) are adjacent if and only if \( L_i R_i S_i x, x S_i R'_i L_{i+1} \in E(H) \). For every \( i \in [t], \)
since (3.16) holds, we have \( \deg_R(z_i) \geq (1 - 2\sqrt{\rho})t \) by the definition of \( G \). Let \( Z' = \{z_{2q+1}, \ldots, z_k\} \) and \( X_0 \) be the set of \( x \in X \) such that \( \deg_R(x, Z') \leq |Z'|/2 \). Then
\[
|X_0|\frac{|Z'|}{2} \leq \sum_{x \in X} \deg_R(x, Z') \leq 2\sqrt{\rho}t \cdot |Z'|,
\]
which implies that \( |X_0| \leq 4\sqrt{\rho}t = 4\sqrt{\rho}\frac{|Y| - \ell}{2k - 2\ell - 1} \leq \frac{4}{3}\sqrt{\rho}|Y| \) (note that \( 2k - 2\ell - 1 \geq k \geq 3 \)).

We now find a perfect matching between \( X \) and \( Z \) as follows.

Step 1: Each \( x \in X_0 \) is matched to some \( z_{2i}, i \in [q] \) such that the corresponding \( Q_{2i} \in F' \) is suitable for \( x \) (thus \( x \) and \( z_{2i} \) are adjacent in \( \Gamma \)) — this is possible because of (3.14) and \( |X_0| \leq \frac{4}{3}\sqrt{\rho}|Y| \).

Step 2: Each of the unused \( z_i, i \in [2q] \) is matched to a vertex in \( X \setminus X_0 \) — this is possible because \( \deg_R(z_i) \geq (1 - 2\sqrt{\rho})t \geq |X_0| + 2q \).

Step 3: Let \( X' \) be the set of the remaining vertices in \( X \). Then \( |X'| = t - 2q = |Z'| \).

Now consider the induced subgraph \( \Gamma' \) of \( \Gamma \) on \( X' \cup Z' \). Since \( \delta(\Gamma') \geq |X'|/2 \), the Marriage Theorem provides a perfect matching in \( \Gamma' \).

The perfect matching between \( X \) and \( Z \) gives rise to the desired Hamilton path of \( H \). \( \Box \)

4. Concluding remarks

Let \( h_d^\ell(k, n) \) denote the minimum integer \( m \) such that every \( k \)-graph \( H \) on \( n \) vertices with minimum \( d \)-degree \( \delta_d(H) \geq m \) contains a Hamilton \( \ell \)-cycle (provided that \( k - \ell \) divides \( n \)). In this paper we determined \( h_d^\ell(k, n) \) for all \( \ell < k/2 \) and sufficiently large \( n \). Unfortunately our proof does not give \( h_d^\ell(k, n) \) for all \( k, \ell \) such that \( k - \ell \) does not divide \( k \) even though we believe that \( h_d^\ell(k, n) = \frac{n}{\ell}\log\frac{n}{\ell} \). In fact, when \( k - \ell \) does not divide \( k \), if we can prove a path-cover lemma similar to Lemma 2.3, then we can follow the proof in [13] to solve the nonextremal case. When \( \ell \geq k/2 \), we cannot define \( \mathcal{Y}_{k, 2\ell} \) so the current proof of Lemma 2.3 fails. In addition, when \( \ell \geq k/2 \), the extremal case becomes complicated as well.

The situation is quite different when \( k - \ell \) divides \( k \). When \( k \) divides \( n \), one can easily construct a \( k \)-graph \( H \) such that \( \delta_{k - 1}(H) \geq \frac{n}{2} - k \) and yet \( H \) contains no perfect matching and consequently no Hamilton \( \ell \)-cycle for any \( \ell \) such that \( k - \ell \) divides \( k \). A construction in [16] actually shows that \( h_{k-1}^\ell(k, n) \geq \frac{n}{2} - k \) whenever \( k - \ell \) divides \( k \), even when \( k \) does not divide \( n \). The exact value of \( h_d^\ell(k, n) \), when \( k - \ell \) divides \( k \), is not known except for \( h_d^3(3, n) = \lfloor n/2 \rfloor \) given in [21]. In the forthcoming paper [9], we determine \( h_d^{k/2}(k, n) \) exactly for even \( k \) and any \( d \geq k/2 \).

Let \( t_d(n, F) \) denote the minimum integer \( m \) such that every \( k \)-graph \( H \) on \( n \) vertices with minimum \( d \)-degree \( \delta_d(H) \geq m \) contains a perfect \( F \)-tiling. One of the first results on hypergraph tiling was \( t_2(n, \mathcal{Y}_{3,2}) = n/4 + o(n) \) given by Kühn and Osthus [14]. The exact value of \( t_2(n, \mathcal{Y}_{3,2}) \) was determined recently by Czygrinow, DeBiasio, and Nagle.
We [8] determined \( t_1(n, \mathcal{Y}_{3,2}) \) very recently. The key lemma in our proof, Lemma 2.8, shows that every \( k \)-graph \( \mathcal{H} \) on \( n \) vertices with \( \delta_{k-1}(\mathcal{H}) \geq (\frac{1}{2k-b} - o(1))n \) either contains an almost perfect \( \mathcal{Y}_{k,b} \)-tiling or is in the extremal case. Naturally this raises a question: what is \( t_{k-1}(n, \mathcal{Y}_{k,b}) \)? Mycroft [17] recently proved a general result on tiling \( k \)-partite \( k \)-graphs, which implies that \( t_{k-1}(n, \mathcal{Y}_{k,b}) = \frac{n}{2k-b} + o(n) \). The lower bound comes from the following construction. Let \( \mathcal{H}_0 \) be the \( k \)-graph on \( n \in (2k-b)^{2k-b} \) vertices such that \( V(\mathcal{H}_0) = A \cup B \) with \( |A| = \frac{n}{2k-b} - 1 \), and \( E(\mathcal{H}_0) \) consists of all \( k \)-sets intersecting \( A \) and some \( k \)-subsets of \( B \) such that \( \mathcal{H}_0[B] \) contains no copy of \( \mathcal{Y}_{k,b} \). Thus, \( \delta_{k-1}(\mathcal{H}_0) \geq \frac{n}{2k-b} - 1 \). Since every copy of \( \mathcal{Y}_{k,b} \) contains at least one vertex in \( A \), there is no perfect \( \mathcal{Y}_{k,b} \)-tiling in \( \mathcal{H}_0 \). We believe that one can find a matching upper bound by the absorbing method (similar to the proof in [2]). In fact, since we already proved Lemma 2.8, it suffices to prove an absorbing lemma and the extremal case.

Acknowledgments

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